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Jan Lang
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Eigenvalues, Embeddings and Generalised Trigonometric Functions

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Preface

The main theme of these notes is the study, from the standpoint of s-numbers, of operators of Hardy type and related Sobolev embeddings. More precisely, let $p,q \in (1,\infty)$ and suppose that I is the interval (a,b), where $-\infty < a < b < \infty$. Maps $T : L_p(I) \to L_q(I)$ of the form

$$(Tf)(x) = v(x) \int_a^x u(t)f(t)dt, \qquad (1)$$

where u and v are prescribed functions satisfying some integrability conditions, are said to be of Hardy type. They are of importance in connection with 'small ball' problems in probability theory [87] and also in the theory of embeddings of Sobolev spaces when the underlying subset Ω of $\mathbb{R}^n$ is a generalised ridged domain, which means crudely that Ω has a central axis (the generalised ridge) that is the image of a tree under a Lipschitz map [42]. In addition, the literature on such maps T has grown to such an extent that the topic has acquired an independent life. Our object is, so far as we are able, to give an account of the present state of knowledge in this area in the hope that it will stimulate further work. In addition to the main theme, topics that arise naturally include the geometry of Banach spaces, generalised trigonometric functions and the p-Laplacian, and we have not hesitated to develop these subsidiary melodies beyond the strict requirements of Hardy operators when the intrinsic interest warranted it. We hope that the resulting contrapuntal effect will appeal to the reader.

Chapter 1 supplies basic information about bases of Banach spaces and such geometric concepts as strict and uniform convexity, uniform smoothness and super-reflexivity. It also gives an account of very recent work (see [44]) on the representation of compact linear operators $S : X \to Y$, where X and Y are reflexive Banach spaces with strictly convex duals. What emerges is the existence of a sequence (x_n) in the unit sphere of X and a sequence (λ_n) of positive numbers in terms of which the action of S can be described and points $x \in X$ represented, under suitable conditions; the λ_n are norms of the restrictions of S to certain subspaces. These results provide an analogue in Banach spaces of the celebrated Hilbert space results of Erhard Schmidt. As a byproduct we have (in Chap. 3) a proof of the existence of an infinite sequence of 'eigenvectors' of the Dirichlet problem for the p-Laplacian in an arbitrary bounded domain in $\mathbb{R}^n$.

The next chapter gives an account of generalised trigonometric functions. To explain what is involved here, let $p \in (1,\infty)$, put

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)}$$

and let $F_p : [0,1] \to \mathbb{R}$ be given by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt.$$

Then the generalised sine function $\sin_p$ is the function defined on $[0,\pi_p/2]$ to be the inverse of F_p and extended to the whole of $\mathbb{R}$ in a natural way so as to be $2\pi_p$-periodic. Plainly $\sin_2 = \sin$. Moreover, p-analogues of the other trigonometric functions may easily be given: for example, $\cos_p$ is defined to be the derivative of $\sin_p$, from which it follows quickly that

$$\left|\sin_p x\right|^p + \left|\cos_p x\right|^p = 1 \text{ for all } x \in \mathbb{R}.$$

After establishing the main properties of these p-functions and some of the identities obtainable by their use, such as a new representation of the Catalan constant, the chapter finishes with a proof of the fact (first given in [9]) that if p is not too close to 1, then the functions $\sin_p(n\pi_p t)$ form a basis in $L_q(0,1)$ for all $q \in (1,\infty)$. The usefulness of such p-functions is underlined in Chap. 3, where it is shown how $\sin_p$ and $\cos_p$ arise naturally in the study of initial- and boundary-value problems for the one-dimensional p-Laplacian on an interval.

Chapter 4 provides necessary and sufficient conditions for the boundedness and compactness of the Hardy operator T of (1) acting between Lebesgue spaces. The norm of T_0, the particular form of T when $u = v = 1$, is determined explicitly and is shown to be attained at functions expressible in terms of generalised trigonometric functions. After this preparation, Chap. 5 is devoted to the s-numbers of T_0, together with the calculation of s-numbers of the basic Sobolev embedding on intervals. We remind the reader that in the theory of s-numbers, to every bounded linear map $S : X \to Y$, where X and Y are Banach spaces, is attached a non-increasing sequence $(s_n(S))_{n\in\mathbb{N}}$ of non-negative numbers with a view to classifying operators according to the behaviour of $s_n(S)$ as $n \to \infty$. The approximation numbers are particularly important examples: the nth approximation number of S is defined to be

$$a_n(S) = \inf \|S - F\|,$$

where the infimum is taken over all linear maps $F : X \to Y$ with rank less than n. These are special cases of the so-called "strict" s-numbers, further examples of which are provided by the Bernstein, Gelfand, Kolmogorov and Mityagin numbers. As might be expected, the results obtained regarding T_0 are especially sharp when $p = q$. In fact, it then turns out that all the strict s-numbers of T_0 coincide, the nth such number $s_n(T_0)$ being given by the formula

$$s_n(T_0) = \frac{(b-a)\gamma_p}{n+1/2}, \text{ where } \gamma_p = \frac{1}{2\pi} p^{1/p'} (p')^{1/p} \sin(\pi/p).$$

Chapter 6 deals with the general case of the operator T given by (1), in which u and v are merely required to satisfy certain integrability conditions. The precision of the results for T_0 is not obtainable for $T : L_p(I) \to L_p(I)$, but it emerges that if $1 < p < \infty$, then again all the strict s-numbers of T coincide, and that this time the asymptotic formula

$$\lim_{n\to\infty} n s_n(T) = \gamma_p \int_a^b |u(t)v(t)|\, dt$$

holds, where $s_n(T)$ denote the common value of the nth strict s-number of T. The cases $p = 1$ and ∞ present particular difficulties, but even then upper and lower estimates for the approximation numbers of T are obtained. The next chapter develops the theme of Chap. 6: it includes the derivation of more precise asymptotic information about the strict s-numbers of T, given additional restrictions on u and v.

So far, knowledge of the behaviour of the s-numbers of T has been obtained only for the case in which T acts from $L_p(I)$ to itself. When T is viewed as a map from $L_p(I)$ to $L_q(I)$ and $p \neq q$, special problems arise and new techniques are required. Chapter 8 deals with this situation and obtains results by consideration of the variational problem of determining

$$\sup_{g \in T(B)} \|g\|_q,$$

where B is the closed unit ball in $L_p(I)$. When $1 < q < p < \infty$, the asymptotic behaviour of the approximation numbers and the Kolmogorov numbers is established: thus

$$\lim_{n\to\infty} n a_n(T) = C(p,q) \left(\int_a^b |u(t)v(t)|^r\, dt \right)^{1/r},$$

where $C(p,q)$ is an explicitly known function of p and q, and $r = 1/q + 1/p'$. Moreover, when $1 < p < q < \infty$, a corresponding formula is shown to hold for the Bernstein numbers of T. In both cases connections are made between the s-numbers of T and ‘eigenvalues’ of the variational problem mentioned above. We stress the key rôle played in the arguments presented in Chaps. 5–8 by the generalised trigonometric functions; Chap. 8 also uses more topological ideas, such as the Borsuk antipodal theorem.

The final chapter extends the discussion of the Hardy operator to the situation in which it acts on spaces with variable exponent, the $L_{p(\cdot)}$ spaces. Here p is a given function with values in $(1,\infty)$: if p is a constant function the space coincides with the usual L_p space. Such spaces have attracted a good deal of interest lately because they occur naturally in various physical contexts and in variational problems involving integrands with non-standard growth properties.

Contents

Basic Notation

$\mathbb{N}$: Set of all natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{Z}$: Set of all integers
$\mathbb{R}$: Set of all real numbers
$\mathbb{R}_+$: Set of all non-negative real numbers
$\mathbb{R}^n$: n-dimensional Euclidean space

If $\Omega \subset \mathbb{R}^n$, then $\overline{\Omega}$ is the closure of Ω, $\overset{o}{\Omega}$ is the interior of Ω, $\partial\Omega$ is the boundary of Ω

$F \sim G$: G is bounded above and below by positive multiples of F independent of any variables occurring in F, G

Chapter 1
Basic Material

Here we remind the reader of some classical definitions and theorems that will be useful later on. Only some proofs of these results are given, but references are provided to works in which detailed expositions of such matters can be found. We also give an account of the more recent theory, developed in [44], of representations of compact linear operators acting between Banach spaces. Applications of this theory will be given in the following chapters.

As a matter of notation throughout the book, the norm on a Banach space X will usually be denoted by $\|\cdot \mid X\|$ or, $\|\cdot\|_X$, depending on the size of the expression X; if no ambiguity is likely we shall simply write $\|\cdot\|$; and $\|\cdot\|_p$ will often be used to denote the norm on an L_p space.

1.1 Bases and Trigonometric Functions

Let X be a Banach space with norm $\|\cdot\|$. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called a (*Schauder*) *basis* of X if, given any $x \in X$, there is a unique sequence $(a_n)_{n\in\mathbb{N}}$ of scalars such that

$$x = \sum_{n=1}^{\infty} a_n x_n; \text{ that is, } \lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} a_n x_n \right\| = 0. \tag{1.1}$$

Given a basis $(x_n)_{n\in\mathbb{N}}$ of X and $N \in \mathbb{N}$, we define a map $P_N : X \to X$ by

$$P_N(x) = \sum_{n=1}^{N} a_n x_n, \; x = \sum_{n=1}^{\infty} a_n x_n \in X. \tag{1.2}$$

It is clear that P_N is linear and that $P_N^2 = P_N$; P_N is a (partial sum) projection. With more effort (see [122], p. 37), it can be shown that P_N is also bounded and $\sup_N \|P_N\| < \infty$. Moreover, $\dim P_N(X) = N$ and $P_N P_M = P_{\min\{M,N\}}$.

If X is a Hilbert space, a basis $(x_n)_{n\in\mathbb{N}}$ of X is called a *Riesz basis* if the map $(a_n) \longmapsto (\sum_{n=1}^{\infty} a_n x_n)$ is an isomorphism of l_2 onto X. This means that there are

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positive constants c, C such that for all $(a_n) \in l_2$,

$$c \sum_{n=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2 .$$

It is plain that any complete orthonormal system in a separable Hilbert space is a Riesz basis. Examples of such systems are the sequence of trigonometric functions $(e^{in\pi x})_{n\in\mathbb{Q}}$ in $L_2(-1,1)$ and the sequence of standard unit vectors in l_2.

Outside the world of Hilbert spaces with its strong geometrical flavour provided by the notion of orthogonality, more effort is often needed to produce examples of bases. For example, when $1 < p < \infty$ a basis of $L_p(-1,1)$ is given by $(e^{in\pi x})_{n\in\mathbb{Z}}$. This follows from a result due to M. Riesz, namely that (see [56], Chap. 12, Sect. 10, p. 106)

$$\lim_{N\to\infty} \left\| f - \sum_{|n|\leq N} f_n e^{in\pi x} \right\|_p = 0, \tag{1.3}$$

for all $f \in L_p(-1,1)$, where $f_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx$ and $\|\cdot\|_p$ is the usual norm on $L_p(-1,1)$; when $p = 1$, (1.3) is false. Given any $f \in L_p(0,1)$, its odd extension to $L_p(-1,1)$ has a unique representation in terms of the $\sin n\pi x$, which means that $(\sin n\pi x)_{n\in\mathbb{N}}$ is a basis of $L_p(0,1)$ when $1 < p < \infty$; a similar argument applies to $(\cos n\pi x)_{n\in\mathbb{N}_0}$. By way of contrast to this non-obvious result, it is trivial that a basis in the sequence space l_p $(1 \leq p < \infty)$ is given by the standard unit vectors.

Let $(x_n)_{n\in\mathbb{N}}$ be a basis in a Banach space X, let $x = \sum_{n=1}^{\infty} a_n x_n \in X$ and for each $n \in \mathbb{N}$ define a functional x_n^* by $\langle x, x_n^* \rangle = a_n$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its dual X^*. Then $x_n^* \in X^*$ and

$$\|x_n\|_X \, \|x_n^*\|_{X^*} \leq 2 \sup_N \|P_N\| . \tag{1.4}$$

The x_n^* are called *biorthogonal functionals* and are uniquely determined by the conditions $\langle x_m, x_n^* \rangle = \delta_{m,n}$. This leads us to the notion of an *unconditional basis*, by which is meant a basis $(x_n)_{n\in\mathbb{N}}$ such that for every $x \in X$ the series $\sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n$ converges unconditionally; that is, $\sum_{n=1}^{\infty} \left\langle x, x_{\sigma(n)}^* \right\rangle x_{\sigma(n)}$ converges whenever σ is a permutation of the natural numbers. Clearly every Riesz basis is an unconditional basis. It is obvious that the standard unit vectors in l_p $(1 \leq p < \infty)$ form an unconditional basis. However, the trigonometric system $(e^{in\pi x})_{n\in\mathbb{Z}}$ is an unconditional basis of $L_p(-1,1)$ only when $p = 2$, although $L_p(-1,1)$ does have an unconditional basis if $1 < p < \infty$ and fails to have one if $p = 1$. For these results we refer to [122], IID. Note also that if $(x_n)_{n\in\mathbb{N}}$ is a basis of a reflexive space X, then the corresponding biorthogonal functionals x_n^* form a basis of X^* (see [33], Chap. IV, Sect. 3, Lemma 1 and Theorem 3); if in addition $\|x_n\| = 1$ for all $n \in \mathbb{N}$, then from (1.4) we see that $(\|x_n^*\|)_{n\in\mathbb{N}} \in l_\infty$.

1.2 Strict and Uniform Convexity

Let X be a Banach space with dual X^*; the value of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^* \rangle_X$ or $\langle x, x^* \rangle$. We recall that X is said to be *strictly convex* if whenever $x, y \in X$ are such that $x \neq y$ and $\|x\| = \|y\| = 1$, and $\lambda \in (0,1)$, then $\|\lambda x + (1-\lambda)y\| < 1$. This simply means that the unit sphere in X does not contain any line segment. Note that an equivalent condition on X is that no sphere of any radius and any centre contains a line segment. For example, if $\mathbb{R}^2$ is equipped with the l_p norm it is strictly convex if $1 < p < \infty$ but not if $p = 1$ or ∞.

Proposition 1.1. *The space X is strictly convex if and only if it is the case that whenever $x, y \in X$ are such that $\|x+y\| = \|x\| + \|y\|$ then either $y = 0$ or $x = \lambda y$ for some $\lambda \geq 0$.*

Proof. Suppose that X is strictly convex and $x, y \in X \setminus \{0\}$ are such that $x \neq y$ and $\|x+y\| = \|x\| + \|y\|$. Then $\|x\| \neq \|y\|$, for otherwise $\left\| \frac{x+y}{2} \right\| = \|x\|$, contradicting the strict convexity of X. However, if $\|y\| < \|x\|$, we put $\lambda = \|y\| / \|x\|$ and observe that

$$1 \geq \left\| \frac{x}{\|x\|} + \lambda \left(\frac{y}{\|y\|} - \frac{x}{\|x\|} \right) \right\| \geq \frac{\|x+y\| - \lambda \|x\|}{\|x\|} = \frac{\|x\| + \|y\| - \lambda \|x\|}{\|x\|} = 1,$$

which means that $x = y/\lambda$. The converse is obvious. □

It is plain that every linear subspace of a strictly convex space is itself strictly convex, with the inherited norm. Here are some of the less trivial properties of such spaces.

Proposition 1.2. *Let X be a Banach space with dual X^*. Then X is strictly convex if and only if given any $x^* \in X^* \setminus \{0\}$, there is at most one $x \in X$ such that $\|x\| = 1$ and $\langle x, x^* \rangle = \|x^*\|$; if X is reflexive, such an x exists.*

Proof. Let X be strictly convex and $x^* \in X^* \setminus \{0\}$. Suppose there are two distinct such points x, say x_1 and x_2. Then if $0 < \lambda < 1$,

$$\begin{aligned} \|x^*\| &= \lambda \langle x_1, x^* \rangle + (1-\lambda) \langle x_2, x^* \rangle = \langle \lambda x_1 + (1-\lambda) x_2, x^* \rangle \\ &\leq \|x^*\| \, \|\lambda x_1 + (1-\lambda) x_2\| < \|x^*\|, \end{aligned}$$

which is absurd. Conversely, suppose that $\|x + \lambda(y-x)\| = 1$ for some $x, y \in X$ with $\|x\| = \|y\| = 1$ and some $\lambda \in (0,1)$. By the Hahn–Banach theorem, there exists $x^* \in X^*$ such that $\langle x + \lambda(y-x), x^* \rangle = 1$ and $\|x^*\| = 1$. Then $(1-\lambda) \langle x, x^* \rangle + \lambda \langle y, x^* \rangle = 1$, and since $|\langle x, x^* \rangle|, |\langle y, x^* \rangle| \leq 1$ we must have $\langle x, x^* \rangle = \langle y, x^* \rangle = 1$. By hypothesis, this implies that $x = y$, and so X is strictly convex.

Now suppose that X is reflexive and let (x_k) be a sequence in X such that $\|x_k\| = 1$ for all $k \in \mathbb{N}$ and $\|x^*\| = \lim_{k \to \infty} \langle x_k, x^* \rangle$. Since X is reflexive, there is a weakly convergent subsequence of (x_k), still denoted by (x_k) for simplicity, with weak limit x, say. Then $\|x\| \leq 1$ and $\langle x, x^* \rangle = \lim_{k \to \infty} \langle x_k, x^* \rangle = \|x^*\|$. □

Proposition 1.3. *Let X be a Banach space. Then X^* is strictly convex if and only if given any $x \in X \backslash \{0\}$, there is a unique $x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Proof. Suppose that X^* is strictly convex. By the Hahn–Banach theorem, there exists $x^* \in X^*$ with $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$. Suppose there exists $y^* \in X^*$, $y^* \neq x^*$, with $\|y^*\| = 1$ and $\langle x, y^* \rangle = \|x\|$. Then if $0 < \lambda < 1$,

$$\|x\| = \lambda \langle x, x^* \rangle + (1-\lambda) \langle x, y^* \rangle \leq \|\lambda x^* + (1-\lambda) y^*\| \, \|x\| < \|x\|,$$

and we have a contradiction. The proof of the converse is similar to that of the corresponding statement in Proposition 1.2. □

Proposition 1.4. *Let K be a convex subset of a strictly convex Banach space X. Then there is at most one element $x \in K$ such that*

$$\|x\| = \inf\{\|y\| : y \in K\}.$$

If, in addition, X is reflexive and K is closed and non-empty, then such a point x exists.

Proof. Suppose there exist $x, y \in K$ with $\|x\| = \|y\| = \inf\{\|z\| : z \in K\}$, $x \neq y$. Let $0 < \lambda < 1$: then $\lambda x + (1-\lambda) y \in K$, $\|\lambda x + (1-\lambda) y\| < \|x\|$ and we have a contradiction.

For the second assertion, let (x_k) be a sequence in K such that $\lim_{k \to \infty} \|x_k\| = l := \inf\{\|y\| : y \in K\}$. By reflexivity, this sequence has a subsequence, still denoted by (x_k) for convenience, such that $x_k \rightharpoonup x$ for some $x \in X$, by which we mean that (x_k) converges weakly to x in X; in fact, $x \in K$ since K is convex and closed, and hence weakly closed. Moreover, $\|x\| \leq \lim_{k \to \infty} \|x_k\| = l$. □

To measure the degree of strict convexity of a Banach space X, we define its *modulus of convexity* $\delta_X : [0,2] \to [0,1]$ by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}. \tag{1.5}$$

This function is introduced so that, given any two distinct points x and y in the closed unit ball B in X, we shall have an idea of how far inside B is the midpoint of the line segment joining x to y. Note that the same function is obtained if the infimum is taken over all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| = \varepsilon$: see [88], Vol. II, p. 60. The Banach space X is called *uniformly convex* if for all $\varepsilon \in (0,2]$, $\delta_X(\varepsilon) > 0$. This means that X is uniformly convex if, given any $\varepsilon \in (0,2]$, $\|x + y\| \leq 2(1 - \delta_X(\varepsilon))$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. It is easy to see that an equivalent condition is that whenever (x_k) and (y_k) are sequences in X such that $\|x_k\| \leq 1, \|y_k\| \leq 1$ $(k \in \mathbb{N})$ and $\lim_{k \to \infty} \|x_k + y_k\| = 2$, then $\lim_{k \to \infty} \|x_k - y_k\| = 0$. Note that δ_X is increasing on $[0,2]$, continuous on $[0,2)$ (not necessarily at 2) and $\delta_X(0) = 0$. Moreover, $\delta_X(2) = 1$ if and only if X is strictly convex.

Plainly every closed linear subspace of a uniformly convex space is uniformly convex when given the inherited norm. Moreover, it is clear that every uniformly convex space is strictly convex: as we shall see later, the converse is false. However, in finite-dimensional spaces strict convexity does imply uniform convexity, for then the fact that $1-\frac{1}{2}\|x+y\|>0$ whenever $\|x\|=\|y\|=1$ and $\|x-y\|=\varepsilon>0$ implies that $\delta_X(\varepsilon)>0$ since closed bounded sets are compact in finite-dimensional spaces. The simplest example of a uniformly convex space is any Hilbert space H, for the parallelogram law enables us to see that

$$\delta_H(\varepsilon)=1-(1-\varepsilon^2/4)^{1/2},$$

which is clearly positive for all $\varepsilon\in(0,2]$; in fact, since

$$(1-x^q)^{1/q}\le 1-x^q/q \quad (0\le x\le 1, 1<q<\infty),$$

we have

$$\delta_H(\varepsilon)\ge\varepsilon^2/8.$$

Note that in the opposite direction, for every Banach space X of dimension at least 2, it is known that

$$\delta_X(\varepsilon)\le\delta_H(\varepsilon)=1-(1-\varepsilon^2/4)^{1/2}\le C\varepsilon^2;$$

see [88], Vol. II, p. 63. In this sense, Hilbert spaces are the 'most' uniformly convex spaces.

If $1<p<\infty$, the sequence space l_p and the Lebesgue space L_p (over any set and with any measure) are uniformly convex. This was established by Clarkson [25], who introduced the notion of uniform convexity and proved the following inequalities (now known as the Clarkson inequalities) that hold for arbitrary elements $f,g\in L_p$:

$$\left(\|(f+g)/2\|_p^{p'}+\|(f-g)/2\|_p^{p'}\right)^{1/p'}\le\left(\frac{\|f\|_p^p+\|g\|_p^p}{2}\right)^{1/p} \text{ if } 1\le p\le 2, \quad (1.6)$$

and

$$\left(\|(f+g)/2\|_p^{p}+\|(f-g)/2\|_p^{p}\right)^{1/p}\le\left(\frac{\|f\|_p^p+\|g\|_p^p}{2}\right)^{1/p} \text{ if } 2\le p\le\infty. \quad (1.7)$$

Here $\|\cdot\|_p$ denotes the norm on L_p. In the cases $2\le p\le\infty$ and $1\le p\le 2$ the inequalities in (1.6) and (1.7) respectively hold in the reversed sense. He used these to obtain lower bounds for $\delta_p:=\delta_{L_p}$ of the form

$$\delta_p(\varepsilon)\ge(\varepsilon/C)^r, \quad (1.8)$$

where $r = p'$ if $1 < p \leq 2$, $r = p$ if $2 \leq p < \infty$, and C is a positive constant that depends on p. These lower bounds are not optimal when $1 < p \leq 2$; however, from them the uniform convexity of L_p when $1 < p < \infty$ follows immediately.

Estimates from below of the modulus of convexity of a Banach space of the form (1.8) are very useful, particularly if the best value of r is known, and so we present below results leading to the optimal value of r for L_p, following the approach of Ball et al. [3]. We begin with a technical lemma.

Lemma 1.1. *Let* $1 < p < \infty$ *and define* $\lambda_p : [0,\infty) \to [0,\infty)$ *by*

$$\lambda_p(t) = (1+t)^{p-1} + |1-t|^{p-1} \; sgn \; (1-t).$$

Then for all $x, y \in \mathbb{R}$,

$$|x+y|^p + |x-y|^p = \sup\{\lambda_p(t)\,|x|^p + \lambda_p(1/t)\,|y|^p : 0 < t < \infty\}$$

if $1 < p \leq 2$; *if* $2 \leq p < \infty$, *the same holds with* sup *replaced by* inf.

Proof. Suppose that $1 < p \leq 2$. By homogeneity and symmetry it is enough to deal with the case $0 < y \leq x = 1$. Put

$$f(t) = \lambda_p(t) + \lambda_p(1/t)y^p, \; 0 < t < \infty,$$

and note that $f(y) = (1+y)^p + (1-y)^p$. Moreover, if $t \neq y$, a routine calculation shows that

$$f'(t) = (p-1)\{1-(y/t)^p\}\left\{(1+t)^{p-2} - |1-t|^{p-2}\right\}.$$

Thus $f'(t) \geq 0$ if $0 < t < y$, $f'(t) \leq 0$ if $y < t < \infty$. It follows that the maximum value of f on $(0,\infty)$ occurs when $t = y$, and the desired inequality follows. The proof when $2 \leq p < \infty$ is similar. □

In [3] it is shown that Clarkson's inequalities may be obtained from this result, which also gives rise to the following theorem, due originally to Hanner [71] and from which the modulus of convexity δ_p can be estimated.

Theorem 1.1. *Suppose that* $1 < p \leq 2$ *and let* $f, g \in L_p$. *Then*

$$\|f+g\|_p^p + \|f-g\|_p^p \geq \left(\|f\|_p + \|g\|_p\right)^p + \left|\|f\|_p - \|g\|_p\right|^p.$$

If $2 \leq p < \infty$ *the inequality is reversed.*

Proof. Suppose that $1 < p \leq 2$. Then by Lemma 1.1,

$$\begin{aligned}\|f+g\|_p^p+\|f-g\|_p^p &= \int \{|f+g|^p+|f-g|^p\} \\ &= \int \sup_{0<t<\infty} \left\{\lambda_p(t)\,|f|^p+\lambda_p(1/t)\,|g|^p\right\} \\ &\geq \sup_{0<t<\infty} \int \left\{\lambda_p(t)\,|f|^p+\lambda_p(1/t)\,|g|^p\right\} \\ &= \sup_{0<t<\infty} \left\{\lambda_p(t)\,\|f\|^p+\lambda_p(1/t)\,\|g\|^p\right\} \\ &= \left(\|f\|_p+\|g\|_p\right)^p+\left|\|f\|_p-\|g\|_p\right|^p,\end{aligned}$$

the final step following from another application of Lemma 1.1. The case $2 \leq p < \infty$ is similar. □

Finally we give a theorem of Ball and Pisier (see [3]).

Theorem 1.2. *Suppose that* $1 < p \leq 2$ *and let* $f, g \in L_p$. *Then*

$$\|f+g\|_p^2+\|f-g\|_p^2 \geq 2\,\|f\|_p^2+2(p-1)\,\|g\|_p^2.$$

If $2 \leq p < \infty$ *the inequality is reversed.*

Proof. Suppose that $1 < p \leq 2$. We use the inequality

$$|a+b|^p+|a-b|^p \geq 2\{a^2+(p-1)b^2\}^{p/2} \;(a,b \in \mathbb{R}) \tag{1.9}$$

which is ascribed to Gross in [3]. Accepting this for the moment, note that by Hölder's inequality,

$$\|f+g\|_p^p+\|f-g\|_p^p \leq 2^{(2-p)/2}\left(\|f+g\|_p^2+\|f-g\|_p^2\right)^{p/2},$$

from which we have, by Theorem 1.1,

$$\begin{aligned}\left(\|f+g\|_p^2+\|f-g\|_p^2\right)^{p/2} &\geq 2^{-(2-p)/2}\left\{\left(\|f\|_p+\|g\|_p\right)^p+\left|\|f\|_p-\|g\|_p\right|^p\right\} \\ &\geq 2^{p/2}\left\{\|f\|_p^2+(p-1)\,\|g\|_p^2\right\}^{p/2},\end{aligned}$$

the final inequality following from (1.9). This gives the theorem when $1 < p \leq 2$; similar arguments work when $2 \leq p < \infty$.

To prove (1.9) it is enough to show that

$$h(t) := (1+t)^p+(1-t)^p-2\{1+(p-1)t^2\}^{p/2} \geq 0,\; 0 \leq t \leq 1. \tag{1.10}$$

Note that if $0 \leq t < 1$,

$$(1+t)^p + (1-t)^p = 2 + p(p-1)t^2 + 2p(p-1)\sum_{n=2}^{\infty} \frac{(3-p)...(2n-1-p)}{(2n)!} t^{2n}$$

and

$$\{1+(p-1)t^2\}^{p/2} = 1 + \frac{p}{2}(p-1)t^2 + \sum_{n=2}^{\infty} \frac{p(2-p)...(2n-2-p)}{n!2^n}(p-1)^n t^{2n}(-1)^{n+1},$$

so that

$$h(t) = 2p(p-1)\sum_{n=2}^{\infty} \frac{(3-p)...(2n-1-p)}{(2n)!} t^{2n} + I,$$

where

$$I = 2\sum_{m=1}^{\infty} \frac{p(2-p)...(4m-2-p)}{(2m+1)!2^{2m+1}} \left\{2(2m+1) - (4m-p)(p-1)t^2\right\}(p-1)^{2m} t^{4m}.$$

Since $2(2m+1) - (4m-p)(p-1)t^2 > 0$ when $0 \leq t < 1$, $h(t) \geq 0$ and (1.10) follows. □

Now these results are applied to give optimal estimates for δ_p, the modulus of convexity of L_p. First suppose that $1 < p \leq 2$. Then from Theorem 1.2 with $u = f+g$ and $v = f-g$ we see that

$$\|u\|_p^2 + \|v\|_p^2 \geq 2\left\|\frac{u+v}{2}\right\|_p^2 + 2(p-1)\left\|\frac{u-v}{2}\right\|_p^2,$$

from which we have

$$\delta_p(\varepsilon) \geq 1 - (1-(p-1)(\varepsilon/2)^2)^{1/2} \geq (p-1)\varepsilon^2/8.$$

In the terminology of [110], according to which given any $r \in [2,\infty)$ a Banach space X is said to be r-uniformly convex if $\delta_X(\varepsilon) \geq (\varepsilon/C)^r$ for some positive constant C, this means that L_p is 2-uniformly convex. When $2 \leq p < \infty$, direct use of Theorem 1.1 shows that $\delta_p(\varepsilon) \geq 1 - (1-(\varepsilon/2)^p)^{1/p} \geq 1 - (1-(\varepsilon/2)^p/p) = (\varepsilon/2)^p/p$, so that L_p is p-uniformly convex. The asymptotic behaviour of δ_p for $1 < p < \infty$ is stated in [88], Vol. II, p. 63; see also [107], p. 237.

To provide other interesting examples of uniformly convex spaces it is convenient to introduce certain spaces of l_p type. Let $I \subset \mathbb{N}$ be non-empty, and for each $i \in I$ let A_i be a Banach space; put $A = \{A_i : i \in I\}$ and let $1 < p < \infty$. Define

$$l_p(A) = \left\{ a = (a_i)_{i \in I} : a_i \in A_i, \left\|a \mid l_p(A)\right\| = \left(\sum_{i \in I} \|a_i \mid A_i\|^p\right)^{1/p} < \infty \right\}.$$

This is a linear space when given the natural definitions of addition and multiplication by scalars; it is routine to verify that it becomes a Banach space when endowed with the norm $\left\|\cdot \mid l_p(A)\right\|$. The following result is due to Day [30].

Theorem 1.3. *The space $l_p(A)$ is strictly convex if and only if each A_i is strictly convex; it is uniformly convex if and only if every A_i is uniformly convex with modulus of convexity δ_i and $\delta(\varepsilon) := \inf_{i\in I} \delta_i(\varepsilon) > 0$ for all $\varepsilon \in (0,2)$.*

Proof. First suppose that every A_i is strictly convex, and let $a,b \in l_p(A)\backslash\{0\}$ be such that $\left\|a+b \mid l_p(A)\right\| = \left\|a \mid l_p(A)\right\| + \left\|b \mid l_p(A)\right\|$. Then

$$\left(\sum_{i\in I} \|a_i + b_i \mid A_i\|^p\right)^{1/p} = \left(\sum_{i\in I} \|a_i \mid A_i\|^p\right)^{1/p} + \left(\sum_{i\in I} \|b_i \mid A_i\|^p\right)^{1/p},$$

and so, by a natural adaptation of the classical method of dealing with the case of equality in Minkowski's inequality, we see that $a = \lambda b$ for some $\lambda > 0$. By Proposition 1.1 it follows that $l_p(A)$ is strictly convex. That strict convexity of $l_p(A)$ implies that of each A_i is obvious.

Now assume that each A_i is uniformly convex and has modulus of convexity δ_i, with $\delta(\varepsilon) := \inf_{i\in I} \delta_i(\varepsilon) > 0$ for all $\varepsilon \in (0,2)$. Let $\varepsilon > 0$, ε small, and suppose initially that $a,b \in l_p(A)$ are such that $\left\|a \mid l_p(A)\right\| = \left\|b \mid l_p(A)\right\| = 1$, $\left\|a-b \mid l_p(A)\right\| > \varepsilon$ and $\|a_i \mid A_i\| = \|b_i \mid A_i\| = \beta_i$ for all $i \in I$; put $\|a_i - b_i \mid A_i\| = \gamma_i$, and note that $\gamma_i \le 2\beta_i$. Then $\|a_i + b_i \mid A_i\| < 2(1-\delta(\gamma_i/\beta_i))\beta_i$ for all $i \in I$, and so

$$\left\|a+b \mid l_p(A)\right\| \le 2\left\{\sum_{i\in I} ((1-\delta(\gamma_i/\beta_i))\beta_i)^p\right\}^{1/p}.$$

Let $E = \{i \in I : \gamma_i/\beta_i > \varepsilon/4\}$ and $F = I\backslash E$. Then

$$1 = \left(\sum_{i\in I} \beta_i^p\right)^{1/p} \ge \left(\sum_{i\in F} \beta_i^p\right)^{1/p} \ge (4/\varepsilon)\left(\sum_{i\in F} \gamma_i^p\right)^{1/p}.$$

Hence

$$\left(\sum_{i\in E} \gamma_i^p\right)^{1/p} = \left(\sum_{i\in I} \gamma_i^p - \sum_{i\in F} \gamma_i^p\right)^{1/p} \ge (\varepsilon^p - (\varepsilon/4)^p)^{1/p} \ge 3\varepsilon/4,$$

so that

$$\alpha := \left(\sum_{i\in E} \beta_i^p\right)^{1/p} \ge 3\varepsilon/8.$$

It follows that

$$\begin{aligned}\left\|a+b \mid l_p(A)\right\| &\leq 2\left\{(1-\delta(\varepsilon/4))^p\sum_{i\in E}\beta_i^p+\sum_{i\in F}\beta_i^p\right\}^{1/p}\\ &\leq 2\{(1-\delta(\varepsilon/4))^p\alpha^p+1-\alpha^p\}^{1/p}\\ &\leq 2\{1-(1-(1-\delta(\varepsilon/4))^p)\alpha^p\}^{1/p}\\ &\leq 2\{1-(1-(1-\delta(\varepsilon/4))^p)(3\varepsilon/8)^p\}^{1/p}=2(1-\delta_0(\varepsilon)),\ \text{say,}\end{aligned}$$

which is an inequality of the required form.

Next suppose that $\left\|a \mid l_p(A)\right\| = \left\|b \mid l_p(A)\right\| = 1$ and that $\left\|a+b \mid l_p(A)\right\| > 2(1-\delta_p(\eta))$, where δ_p is the modulus of convexity of $l_p(A)$ and $\eta \in (0,2]$. Then

$$2(1-\delta_p(\eta)) \leq \left(\sum_{i\in I}(\|a_i \mid A_i\|+\|b_i \mid A_i\|)^p\right)^{1/p} \leq 2,$$

and since the sequences $(\|a_i \mid A_i\|)$ and $(\|b_i \mid A_i\|)$ are in l_p, we see that

$$\left(\sum_{i\in I}|\|a_i \mid A_i\|-\|b_i \mid A_i\||^p\right)^{1/p} < \eta.$$

Let $c=(c_i)=(b_i\|a_i \mid A_i\|/\|b_i \mid A_i\|)$, so that $\|c_i \mid A_i\|=\|a_i \mid A_i\|$ for all $i \in I$ and $\left\|b-c \mid l_p(A)\right\| = \left(\sum_{i\in I}|\|b_i \mid A_i\|-\|a_i \mid A_i\||^p\right)^{1/p} < \eta$. Thus by the earlier part of our discussion, $\left\|a-c \mid l_p(A)\right\| \leq \varepsilon/2$ if $\left\|a+c \mid l_p(A)\right\| \geq 2(1-\delta_0(\varepsilon/2))$; moreover, $\left\|a+c \mid l_p(A)\right\| \geq \left\|a+b \mid l_p(A)\right\| - \left\|c-b \mid l_p(A)\right\| > 2(1-\delta_p(\eta)-\eta/2)$. Choose $\eta \in (0,\varepsilon/2)$ so that $\delta_p(\eta)+\eta/2 < \delta_0(\varepsilon/2)$. Then $\left\|a+b \mid l_p(A)\right\| > 2(1-\delta_p(\eta))$ implies that $\left\|a+c \mid l_p(A)\right\| > 2(1-\delta_0(\varepsilon/2))$, which in turn implies that $\left\|a-c \mid l_p(A)\right\| < \varepsilon/2$. Thus $\left\|a-b \mid l_p(A)\right\| \leq \left\|a-c \mid l_p(A)\right\| + \left\|b-c \mid l_p(A)\right\| < \varepsilon$. Set $\delta_p'(\varepsilon)=\delta_p(\eta)>0$: then $\left\|a-b \mid l_p(A)\right\| < \varepsilon$ if $\left\|a \mid l_p(A)\right\| = \left\|b \mid l_p(A)\right\| = 1$ and $\left\|a+b \mid l_p(A)\right\| > 2(1-\delta_p'(\varepsilon))$. This establishes the uniform convexity of $l_p(A)$.

Finally, suppose that $l_p(A)$ is uniformly convex. Consideration of the points $a = (\delta_{ij}) \in A$, where δ_{ij} is the Kronecker delta, shows that each A_j is uniformly convex. □

If each $A_i = L_{p_i}$, then since by our earlier results the modulus of convexity of L_{p_i} satisfies

$$\delta_{L_{p_i}}(\varepsilon) \geq \left(\frac{p_i-1}{8}\right)\varepsilon^2 \text{ if } 1<p_i\leq 2, \text{ and } \delta_{L_{p_i}}(\varepsilon) \geq (\varepsilon/2)^{p_i}/p_i \text{ if } 2<p_i<\infty,$$

it follows that $l_p(A)$ is uniformly convex if and only if the sequence $\{p_i\}$ is bounded away from 1 and ∞.

As a further immediate application of this result we deal with Sobolev spaces. Let Ω be an open subset of $\mathbb{R}^n$, let $p \in (1,\infty)$ and suppose that $k \in \mathbb{N}$. The Sobolev space $W_p^k(\Omega)$ is the linear space of all (equivalence classes of) functions $u \in D^\alpha u$ such that for all $\alpha = (\alpha_j) \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + ... + \alpha_n \le k$, the distributional derivative $D^\alpha u := \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$ belongs to $L_p(\Omega)$. Endowed with the norm

$$\left\| u \mid W_p^k(\Omega) \right\| := \left(\sum_{|\alpha| \le k} \left\| D^\alpha u \mid L_p(\Omega) \right\|^p \right)^{1/p}$$

it becomes a Banach space, which is isometrically isomorphic to a closed linear subspace V of the product of $N := \#\{\alpha \in \mathbb{N}_0^n : |\alpha| \le k\}$ copies of $L_p(\Omega)$, the isomorphism being established by the map $u \longmapsto (D^\alpha u)_{|\alpha| \le k}$. With $I = \{1, 2, ..., N\}$ and $A_j = L_p(\Omega)$ for all $j \in I$, it is plain that this product is just $l_p(A)$, and in view of Theorem 1.3, $l_p(A)$ is uniformly convex. Thus V is uniformly convex, and therefore so is $W_p^k(\Omega)$. We have thus proved

Corollary 1.1. *For all $p \in (1,\infty)$ and all $k \in \mathbb{N}$, the Sobolev space $W_p^k(\Omega)$ is uniformly convex.*

Remark 1.1. (i) In [29] and [30] the spaces $l_p(A)$ are used to show that strict convexity does not imply uniform convexity, and indeed that there are separable, reflexive, strictly convex Banach spaces that are not isomorphic to any uniformly convex space. When $A = \{l_{p_j} : j \in \mathbb{N}\}$, $l_p(A)$ is isomorphic to a uniformly convex space if and only if the sequence (p_j) is bounded away from 1 and ∞. A result more general than Theorem 1.3 is given in [31], the idea being to replace the l_p structure of the space $l_p(A)$ by means of a more general norm.

(ii) In [30] an analogue of Theorem 1.3 is given for spaces of Lebesgue type. Let $1 < p < \infty$, let X be a Banach space, let Ω be an open subset of $\mathbb{R}^n$ and denote by $L_p(X)$ the space of all functions $f : \Omega \to X$ such that

$$\left\| f \mid L_p(X) \right\| := \left(\int_\Omega \left\| f(x) \mid X \right\|^p dx \right)^{1/p} < \infty.$$

Then [30] shows that $L_p(X)$, endowed with the norm $\left\| \cdot \mid L_p(X) \right\|$, is uniformly convex if and only if X is uniformly convex. The same holds with any measure space instead of Ω and Lebesgue measure. An application of this result will be made in Chap. 3.

We now give some useful properties of uniformly convex spaces.

Proposition 1.5. *Let X be uniformly convex and let (x_k) be a sequence in X that converges weakly to $x \in X$, with $\|x_k\| \to \|x\|$. Then $\|x_k - x\| \to 0$.*

Proof. As the result is trivial if $x = 0$ we assume that $x \neq 0$; we may plainly also assume that for all $k \in \mathbb{N}$, $x_k \neq 0$. Put $\xi_k = 1 - \|x\| / \|x_k\|$; $\xi_k \to 0$ as $k \to \infty$. Set

$y_k = x_k / \|x_k\|$, $y = x / \|x\|$ and note that $y_k = (1-\xi_k)x_k / \|x\|$ converges weakly to y. Moreover, $\|y_k\| = \|y\| = 1$. By the Hahn–Banach theorem, there exists $y^* \in X^*$ such that $\langle y, y^* \rangle_X = 1 = \|y^*\|$. Hence

$$2 \geq \|y_k + y\| \geq \langle y_k + y, y^* \rangle_X \to 2 \langle y, y^* \rangle_X = 2,$$

so that $\lim_{k\to\infty} \|y_k + y\| = 2$. Since X is uniformly convex, $\lim_{k\to\infty} \|y_k - y\| = 0$, and as $x_k - x = \|x_k\| y_k - \|x\| y$, it follows easily that $\|x_k - x\| \to 0$. □

Theorem 1.4. *Every uniformly convex space is reflexive.*

Proof. This classical result is due to D.P. Milman [98]; here we give the short proof contained in [88], Vol. II, Prop. 1.e.3. Suppose that X is uniformly convex, let $J : X \to X^{**}$ be the canonical map, let B, B^{**} be the closed unit balls in X, X^{**} respectively and denote by σ the weak topology $\sigma(X^{**}, X^*)$. Let $x^{**} \in X^{**}$, $\|x^{**}\| = 1$. Then since $J(B)$ is σ-dense in B^{**} (see [15], Chap. IV, Sect. 5), there is a generalised sequence $\{x_\alpha\}_{\alpha\in A}$ $(x_\alpha \in B)$ such that $Jx_\alpha \xrightarrow{\sigma} x^{**}$. Since $Jx_\alpha + Jx_\beta \xrightarrow{\sigma} 2x^{**}$, the weak lower-semicontinuity of the norm shows that $\lim_{\alpha,\beta} \|Jx_\alpha + Jx_\beta\| = 2$, and hence

$$\lim_{\alpha,\beta} \|x_\alpha + x_\beta\| = 2. \tag{1.11}$$

As X is uniformly convex, it follows from (1.11) that $\lim_{\alpha,\beta} \|x_\alpha - x_\beta\| = 0$, so that for some $x \in B$, $\lim_\alpha \|x_\alpha - x\| = 0$. Thus $\lim_\alpha \|Jx_\alpha - Jx\| = 0$ and hence $x^{**} = Jx : X$ must be reflexive. □

Together with Proposition 1.4 this gives

Proposition 1.6. *Let K be a closed, convex, non-empty subset of a uniformly convex space X. Then K has a unique element of minimal norm.*

Now let X be uniformly convex and let $K \subset X$ be closed, convex and non-empty. Given any $x \in X$, the set $K - \{x\} := \{y - x : y \in K\}$ is closed and convex and so has a unique element $w(x)$ of minimal norm, by Proposition 1.6. Clearly $w(x) = z(x) - x$ for some unique $z(x) \in K$ and

$$\|z(x) - x\| = \inf\{\|y - x\| : y \in K\} = d(x, K).$$

In other words, there is a unique point $P_K x := z(x)$ of K such that $d(x,K) = \|x - P_K x\|$. The mapping $P_K : X \to K$ is called the *projection map of X onto K*; plainly $P_K x = x$ if and only if $x \in K$.

Proposition 1.7. *Let K be a closed, convex, non-empty subset of a uniformly convex space X. Then the projection map P_K is continuous.*

Proof. Suppose the result is false. Then there exist $x \in X$, a sequence (x_n) in X and $\varepsilon > 0$ such that $\lim_{n\to\infty} x_n = x$ and $\|P_K x_n - P_K x\| \geq \varepsilon$ for all $n \in \mathbb{N}$. Since $|d(x,K) - d(x_n,K)| \leq \|x_n - x\|$, it follows that

$$|\|x_n - P_K x_n\| - \|x - P_K x\|| \leq \|x_n - x\| \to 0 \text{ as } n \to \infty.$$

As $(P_K x_n)$ is bounded, it has a weakly convergent subsequence, still denoted by $(P_K x_n)$ for convenience, with weak limit z, say. Since K is closed and convex, it is weakly closed and so $z \in K$. Moreover, $x_n - P_K x_n \rightharpoonup x - z$ and

$$\|x - z\| \leq \liminf_{n \to \infty} \|x_n - P_K x_n\| = \|x - P_K x\|,$$

which implies that $z = P_K x$. Hence

$$x_n - P_K x_n \rightharpoonup x - P_K x \text{ and } \|x_n - P_K x_n\| \to \|x - P_K x\|.$$

By Proposition 1.5, the uniform convexity of X now implies that $x_n - P_K x_n \to x - P_K x$, so that $P_K x_n \to P_K x$: contradiction. □

If X is a Hilbert space H with inner product $(\cdot,\cdot)$, this last result can be sharpened, for then

$$\|P_K x - P_K y\| \leq \|x - y\| \text{ for all } x, y \in H.$$

To establish this, note that for all $w \in K$ and all $x \in H$,

$$\|x - P_K x\| \leq \|x - w\|.$$

Thus if $\lambda \in (0,1)$,

$$\|x - P_K x\|^2 \leq \|x - (1-\lambda) P_K x - \lambda P_K y\|^2 = \|x - P_K x + \lambda (P_K x - P_K y)\|^2.$$

Hence

$$0 \leq 2 \text{ re } (x - P_K x, P_K x - P_K y) + \lambda \|P_K x - P_K y\|^2.$$

Similarly,

$$0 \leq 2 \text{ re } (y - P_K y, P_K y - P_K x) + \lambda \|P_K x - P_K y\|^2.$$

Addition of these inequalities gives

$$\text{re } (x - y + P_K y - P_K x, P_K x - P_K y) \geq -\lambda \|P_K x - P_K y\|^2.$$

Letting $\lambda \to 0$ we see that

$$\text{re } (x - y, P_K y - P_K x) \geq \|P_K x - P_K y\|^2,$$

from which the result follows easily.

A map $\mu : [0,\infty) \to [0,\infty)$ that is continuous, strictly increasing and satisfies $\mu(0) = 0$, $\lim_{t \to \infty} \mu(t) = \infty$, is called a *gauge function*. A map J from a Banach space X to 2^{X^*}, the set of all subsets of X^*, is said to be a *duality map on X with gauge function μ* if

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x^*\| \|x\|, \|x^*\| = \mu(\|x\|)\}.$$

Note that by the Hahn–Banach theorem, for each $x \in X$ the set $J(x)$ is non-empty. It is also convex. To justify this, let $x^*, y^* \in J(x)$ and $\lambda \in (0,1)$. Put $z^* = \lambda x^* + (1-\lambda)y^*$ and observe that

$$\langle x, z^* \rangle = \lambda \langle x, x^* \rangle + (1-\lambda) \langle x, y^* \rangle = \mu(\|x\|) \|x\|.$$

Hence $\|z^*\| \geq \mu(\|x\|)$. However,

$$\|z^*\| \leq \lambda \|x^*\| + (1-\lambda) \|y^*\| = \mu(\|x\|)$$

and so $\|z^*\| = \mu(\|x\|)$. Thus $z^* \in J(x)$ and the proof is complete.

Let X be a Banach space with strictly convex dual X^* and let J be a duality map on X with gauge function μ. Then for each $x \in X$, the set $J(x)$ consists of precisely one point. In fact, for each $x \in X$, the points in $J(x)$ lie on the sphere in X^* with centre 0 and radius $\mu(\|x\|)$. If $J(x)$ contained two distinct points, the midpoint of the line segment joining them would be in the convex set $J(x)$, which is impossible as X^* is strictly convex.

In view of this result, we shall regard a duality map J on X as a map from X to X^* when X^* is strictly convex. It is known (see [19], pp. 42–43 and [94], p. 176) that if X and X^* are strictly convex, then J is an injective map of X onto X^* that is strictly monotone (that is, $\langle x-y, Jx-Jy \rangle > 0$ if $x, y \in X, x \neq y$); it is weakly continuous in the sense that if $x_k \to x_0$ in X, then Jx_k converges weak* in X^* to Jx_0 (the convergence is strong if, in addition, X^* is uniformly convex); the map $J^{-1} : X^* \to X$ is a duality map on X^* with gauge function μ^{-1} if, in addition, X is reflexive, X^{**} being identified with X. When X is a Hilbert space, so that X^* may be identified with X, the most natural duality map on X is the identity map, corresponding to the gauge function μ with $\mu(t) = t$.

When $1 < p < \infty$ and $\mu(t) = t^{p-1}$, it may be checked that the duality map J on $L_p(\Omega)$ (where Ω is, say, a domain in $\mathbb{R}^n$) with gauge function μ is given by

$$J(u) = |u|^{p-2} u. \tag{1.12}$$

The duality map J on l_p with the same gauge function μ is defined by $J((x_k)) = (|x_k|^{p-2} x_k)$. Duality maps on Sobolev spaces will be discussed later.

Next we turn to differentiability. Let F be a real-valued function on a Banach space X. We say that F is *Gâteaux-differentiable* at $x_0 \in X$ if there exists $x^* \in X^*$ such that

$$\langle h, x^* \rangle = \lim_{t \to 0} \frac{F(x_0 + th) - F(x_0)}{t} \text{ for all } h \in X. \tag{1.13}$$

The limit above is called the *derivative of F in the direction h*; the functional x^* is denoted by grad $F(x_0)$ and will be referred to as the *gradient* or *Gâteaux derivative* of F at x_0. Another notion of differentiability of F is that of Fréchet: F is said to be *Fréchet-differentiable* at x_0 if there exists a functional $F'(x_0) \in X^*$, called the (Fréchet) derivative of F at x_0, such that

$$\frac{|F(x_0+h) - F(x_0) - \langle h, F'(x_0)\rangle|}{\|h\|} \to 0 \text{ as } \|h\| \to 0. \tag{1.14}$$

Clearly Fréchet-differentiability implies Gâteaux-differentiability, with equality of grad $F(x_0)$ and $F'(x_0)$; in the reverse direction, it can be shown that if grad $F(x)$ exists throughout some neighbourhood of x_0 and is continuous at x_0, then F is Fréchet-differentiable at x_0 and grad $F(x_0) = F'(x_0)$. As Gâteaux derivatives are often easier to calculate than Fréchet derivatives, this result is quite useful.

The case when $F(x) = \|x\|$ $(x \in X)$ is of particular interest.

Proposition 1.8. *Let X be a Banach space with norm $\|\cdot\|$. Then $\|\cdot\|$ is Gâteaux-differentiable on $X\backslash\{0\}$ if and only if X^* is strictly convex.*

Proof. For the moment we suppose that X is real. Let $x, h \in X$. If $0 < s < t$, then the convexity of $y \mapsto \|x+y\|$ implies that

$$\|x+sh\| - \|x\| = \left\|x + \frac{s}{t}th + (1-\frac{s}{t})\cdot 0\right\| - \|x\| \leq \frac{s}{t}\|x+th\| - \|x\|,$$

which shows that $t \mapsto t^{-1}(\|x+th\| - \|x\|)$ is monotone increasing in $t > 0$. Moreover, $t^{-1}(\|x+th\| - \|x\|)$ is bounded below, for use of the triangle inequality shows that

$$t^{-1}(\|x+th\| - \|x\|) \geq -\|h\|.$$

Hence $\phi_+(h) := \lim_{t\to 0+} t^{-1}(\|x+th\| - \|x\|)$ exists for all $x, h \in X$, as does $\phi_-(h) := \lim_{t\to 0+} t^{-1}(\|x\| - \|x-th\|)$. Since

$$\|x+th\| - \|x\| \geq \|x\| - \|x-th\|,$$

we see that $\phi_-(h) \leq \phi_+(h)$.

If $x^* \in X^*$ is such that $\|x^*\| = 1$ and $\langle x, x^*\rangle = \|x\|$, then

$$\langle h, x^*\rangle = \frac{\langle x - x + th, x^*\rangle}{t} \geq \frac{\|x\| - \|x-th\|}{t},$$

and so $\phi_-(h) \leq \langle h, x^*\rangle$; similarly, $\langle h, x^*\rangle \leq \phi_+(h)$. Put

$$M^*(x) = \{x^* \in X^* : \phi_-(h) \leq \langle h, x^*\rangle \leq \phi_+(h) \text{ for all } h \in X\}.$$

We claim that

$$M^*(x) = \{x^* \in X^* : \|x^*\| = 1 \text{ and } \langle x, x^*\rangle = \|x\|\}.$$

To justify this, let $x^* \in M^*(x)$ and observe that $\phi_-(x) \leq \langle x, x^* \rangle \leq \phi_+(x)$, while clearly $\phi_-(x) = \phi_+(x) = \|x\|$. Hence $\langle x, x^* \rangle = \|x\|$ and also $\|x^*\| = 1$. The claim is justified.

Now suppose that $\|\cdot\|$ is Gâteaux-differentiable on $X \backslash \{0\}$ and let $x \in X \backslash \{0\}$. Then the functionals ϕ_- and ϕ_+ introduced above coincide, which implies that $\langle h, x^* \rangle = \phi_-(h)$ for all $h \in X$ and hence that $M^*(x)$ has exactly one element. By Proposition 1.3, X^* is strictly convex.

Conversely, suppose that X^* is strictly convex and let $x \in X \backslash \{0\}$. Then $M^*(x)$ consists of a single element, x^* say. We claim that this implies that $x^* = \phi_- = \phi_+$. For fixed $y_0 \in X$ define a functional x_1^* on the space spanned by y_0 according to the prescription $\langle \lambda y_0, x_1^* \rangle = \lambda \phi_+(y_0)$. If $\lambda \geq 0$, $\phi_+(\lambda y_0) = \lambda \phi_+(y_0)$, while if $\lambda < 0$,

$$\lambda \phi_+(y_0) = -|\lambda| \phi_+(y_0) = |\lambda| \phi_-(-y_0) \leq |\lambda| \phi_+(-y_0) = \phi_+(\lambda y_0).$$

Hence $\langle \lambda y_0, x_1^* \rangle \leq \phi_+(\lambda y_0)$ for all $\lambda \in \mathbb{R}$. Moreover, ϕ_+ is subadditive, since for $0 < t \leq s$,

$$\begin{aligned} \phi_+(y_1 + y_2) &\leq (t/2)^{-1} \left\{ \left\| x + \frac{t}{2} y_1 + \frac{t}{2} y_2 \right\| - \|x\| \right\} \\ &= t^{-1} \{ \|2x + t y_1 + t y_2\| - 2\|x\| \} \\ &\leq t^{-1} \{ \|x + t y_1\| - \|x\| \} + s^{-1} \{ \|x + s y_2\| - \|x\| \}. \end{aligned}$$

It follows from the Hahn–Banach theorem that there is a linear extension $x_0^* \in X^*$ of x_1^* such that $\langle y, x_0^* \rangle \leq \phi_+(y)$ on X. Thus $|\langle y, x_0^* \rangle| \leq \|y\|$ for all $y \in X$ and so $x_0^* \in X^*$. Moreover, $\langle -y, x_0^* \rangle \geq -\phi_+(y) = \phi_-(y)$ on X : we conclude that $x_0^* = x^*$. Repetition of this argument with ϕ_- instead of ϕ_+ shows that $\phi_-(y) = \langle y, x^* \rangle = \phi_+(y)$ for all $y \in X$, which means that $\|\cdot\|$ is Gâteaux-differentiable on $X \backslash \{0\}$.

When X is complex the arguments are similar, with re x^* instead of x^*. □

Remark 1.2. (i) Proposition 1.8 leads to the description as *smooth* those spaces X for which X^* is strictly convex; the arguments above show that for such spaces X, if $x \in X \backslash \{0\}$, then grad $\|x\|$ is the unique element $x^* \in X^*$ such that $\langle x, x^* \rangle = \|x\|$ and $\|x^*\| = 1$. This implies that the map $J : X \to X^*$ given by $Jx = \mu(\|x\|)\text{grad}\,\|x\|$ $(x \neq 0)$, $J0 = 0$, is the duality map on X with gauge function μ.

(ii) If X^* is uniformly convex and X is real, it can be shown (see, for example, [96], Lemma II.5.7) that grad $\|x\|$ is the Fréchet derivative of $\|\cdot\|$ at x for each $x \in X \backslash \{0\}$.

The Gâteaux derivative of the norm is quite easy to determine directly in some important cases. For example, suppose that $1 < p < \infty$ and consider the sequence space l_p, with norm $\|x\| = (\sum_1^\infty |x_k|^p)^{1/p}$. It is easy to see that as $t \to 0$, then for all $x, h \in X$ with $x \neq 0$,

$$t^{-1} \sum_1^\infty \{ |x_k + t h_k|^p - |x_k|^p \} \to p \sum_1^\infty |x_k|^{p-1} \text{ sgn } x_k \cdot h_k.$$

From this we have

$$\text{grad } \|x\| = p^{-1}\|x\|^{1-p} \text{ grad } \|x\|^p = \left(\|x\|^{1-p}|x_k|^{p-1} \text{ sgn } x_k\right), x \neq 0. \quad (1.15)$$

In a similar way it can be shown that the Gâteaux derivative of the norm $\|x\| = \left(\int_\Omega |x(\xi)|^p d\xi\right)^{1/p}$ on $L_p(\Omega)$, where Ω is, for example, an open subset of $\mathbb{R}^n$, is given by

$$(\text{grad } \|x\|)(\xi) = \|x\|^{1-p}|x(\xi)|^{p-1}\text{sgn } x(\xi), \xi \in \Omega. \quad (1.16)$$

Of course, (1.15) and (1.16) also follow immediately from our earlier calculations of duality maps for l_p and $L_p(\Omega)$.

Further illustrations of the usefulness of Proposition 1.8 can be given by means of polar sets. Given any closed linear subspace M of X, we write

$$M^0 = \{x^* \in X^* : \langle x, x^*\rangle_X = 0 \text{ for all } x \in M\},$$

and call this the polar set of M; similarly, if N is a closed linear subspace of X^*, we set

$${}^0N = \{x \in X : \langle x, x^*\rangle_X = 0 \text{ for all } x^* \in N\}.$$

Corresponding notation is used with respect to linear subspaces of other Banach spaces and their duals. The polar set M^0 of any closed linear subspace M of X is isometrically isomorphic to $(X/M)^*$: in fact, with the canonical map of X onto X/M denoted by ϕ, the adjoint ϕ^* of ϕ is an isometric isomorphism of $(X/M)^*$ onto M^0 (see, for example, [15], Chap. IV, Section 5, Prop. 9). Note also that if X is reflexive, then $(M^0)^*$ is isometrically isomorphic to X/M and X/M is reflexive.

Proposition 1.9. *Let M be a closed linear subspace of a reflexive Banach space X. If X is strictly convex, so are M and X/M; if X^* is strictly convex, so are $(X/M)^*$ and M^0.*

Proof. Suppose first that X is strictly convex. Then clearly so is M. Since X is reflexive, the norm on X^*, and hence that on M^0 and thus on $(X/M)^*$ (which is isometrically isomorphic to M^0) is Gâteaux-differentiable at all non-zero points. Hence by Proposition 1.8, the reflexive space X/M is strictly convex. If X^* is strictly convex, then so is its closed subspace M^0, and as $(X/M)^*$ is isometrically isomorphic to M^0, the result follows. □

Remark 1.3. In [32] the relationship between uniform convexity of a Banach space X and that of its quotient spaces is analysed. Let $\mathscr{M}$ be the family of all closed linear subspaces $M \neq \{0\}$ of X. It turns out that X is uniformly convex if and only if all the quotient spaces X/M $(M \in \mathscr{M})$ are uniformly convex, with a common modulus of convexity; the same holds if $\mathscr{M}$ is replaced by $\{M \in \mathscr{M} : \dim(X/M) = 2\}$. This paper also discusses the interaction between uniform convexity and dual spaces. To explain this, we introduce the modulus of smoothness of X : this is the function $\rho_X : (0,\infty) \to [0,\infty)$ defined by

$$\rho_X(\varepsilon) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \varepsilon\right\}.$$

Since

$$2\|x\| = \|(x+y)+(x-y)\| \leq \|x+y\| + \|x-y\|$$

it follows that $\rho_X(\varepsilon) \geq 0$.

The space X is called *uniformly smooth* if

$$\lim_{\varepsilon\to 0} \rho_X(\varepsilon)/\varepsilon = 0.$$

Note that in the definition of $\rho_X(\varepsilon)$ the supremum may be taken over all $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq \varepsilon$ without affecting the outcome. Uniform smoothness means that if x and y are on the unit sphere and lie in a narrow cone, then $\|x+y\|$ is close to 2; loosely speaking, the unit ball of X does not have any corners. Hilbert spaces H are the 'most' uniformly smooth spaces in the sense that

$$\rho_X(\varepsilon) \geq \rho_H(\varepsilon) \text{ for all } \varepsilon > 0.$$

Observe also that X is uniformly smooth if and only if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \text{ exists, uniformly on } \{(x,y) : \|x\| = \|y\| = 1\}.$$

The main result of [32] is that X is uniformly convex (respectively, uniformly smooth) if and only if X^* is uniformly smooth (respectively, uniformly convex). For a discussion of this and related notions we refer to [33] (Chap. VII, Sect. 2), [3] and [107], 5.5.2. Further insight may be obtained by means of the concept of super-reflexivity. To explain this we need some terminology: given Banach spaces X and Y, the space X is said to be *finitely representable in* Y if, for each finite-dimensional subspace X_n of X and each $\lambda > 1$, there is an isomorphism T_n of X_n into Y for which

$$\lambda^{-1} \|x\|_X \leq \|T_n x\|_Y \leq \lambda \|x\|_X \text{ for all } x \in X_n.$$

A Banach space X is called *super-reflexive* if no nonreflexive Banach space is finitely representable in X. Each of the following properties is equivalent to super-reflexivity of X (see, for example, [107], 5.5.2.4) :

(a) X^* is super-reflexive.
(b) X has an equivalent norm with respect to which it is uniformly convex.
(c) X^* has an equivalent norm with respect to which it is uniformly convex.
(d) X has an equivalent norm with respect to which it is uniformly smooth.

Given a super-reflexive Banach space X with a Schauder basis $(x_n)_{n\in\mathbb{N}}$, where $\|x_n\|_X = 1$ for all $n \in \mathbb{N}$, it can be shown (see [76] and [70]; see also [61], Chap. 9, Theorem 9.25) that there exist $p, q \in (1,\infty)$ and $K > 0$ such that for every $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$,

$$K^{-1}\left(\sum_{n=1}^{\infty}|\alpha_n|^q\right)^{1/q} \le \|x\|_X \le K\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p}.$$

In particular, if X is uniformly convex, then there exists $q \in (1,\infty)$ such that for every $x \in X$ the biorthogonal functionals x_n^* corresponding to the x_n satisfy $(x_n^*(x)) \in l_q$.

We see from the proof of Proposition 1.8 that given any Banach space X and any $x, h \in X$, then

$$(x,h)_+ := \|x\| \lim_{t\to 0+} t^{-1}(\|x+th\| - \|x\|) \text{ and } (x,h)_- := \lim_{t\to 0+} t^{-1}(\|x\| - \|x-th\|) \tag{1.17}$$

both exist. Now suppose that $\|\cdot\|$ is Gâteaux-differentiable on $X\backslash\{0\}$. Then $(x,h)_+ = (x,h)_-$ for all $x,h \in X$; we define $(x,h)_X$ to be this common value and call it the *semi-inner product* of x and h. Thus

$$(x,h)_X = \|x\| \langle h, \text{grad } \|x\|\rangle_X, \text{ for } x,h \in X,\ x \neq 0. \tag{1.18}$$

We define $(0,h)_X = 0$ for all $h \in X$. Note that $(x,h)_X$ depends linearly on h and that $(x,x)_X = \|x\|^2$; however, in general, $(x,h)_X \neq (h,x)_X$. By way of illustration we note that when $X = L_p(\Omega)$ and $1 < p < \infty$,

$$(x,h)_X = \|x\|^{2-p}\int_\Omega |x(\xi)|^{p-2}\overline{x(\xi)}h(\xi)d\xi. \tag{1.19}$$

This leads us to the notion of orthogonality of elements of a Banach space. Following James [75], we give

Definition 1.1. Let X be a Banach space and let $x,h \in X$. Then x is said to be orthogonal to h, written $x \perp h$, if

$$\|x+\lambda h\| \ge \|x\| \text{ for all } \lambda \in \mathbb{R}; \tag{1.20}$$

x is orthogonal to a subset H of X if $x \perp h$ for all $h \in H$. Orthogonality of subsets M_1, M_2 of X, written $M_1 \perp M_2$, is defined in the obvious way.

Note that in general orthogonality is not symmetric, that is, $x \perp h$ does not imply $h \perp x$. Indeed (see [75]), if X is strictly convex with $\dim X \ge 3$, then if orthogonality is symmetric, X must be an inner-product space.

The linkage between orthogonality and semi-inner products is given by the next result, contained in [75].

Theorem 1.5. *Let X be a Banach space with norm $\|\cdot\|$ that is Gâteaux-differentiable at every non-zero point, and let $x,h \in X$. Then $x \perp h$ if and only if $(x,h)_X = 0$. Moreover, given $x \in X\backslash\{0\}$ and $y \in X$, there is precisely one $\lambda \in \mathbb{R}$ such that $x \perp \lambda x + y$ and this is given by*

$$\langle y, grad\ \|x\|\rangle = -\lambda\,\|x\|. \tag{1.21}$$

Finally we mention decompositions of X and X^*, involving this notion of orthogonality, given by Alber [2]. We say that X is the James orthogonal sum of closed linear subspaces M_1, M_2, written $X = M_1 \uplus M_2$, if (a) each $x \in X$ has a unique decomposition $x = m_1 + m_2$, where $m_1 \in M_1$, $m_2 \in M_2$; (b) $M_1 \cap M_2 = \{0\}$; (c) $M_1 \perp M_2$. His result is the following

Theorem 1.6. *Let M be a closed linear subspace of a uniformly convex and uniformly smooth Banach space X; denote by $J_X : X \to X^*$ the duality map with gauge function μ given by $\mu(t) = t$ $(t \geq 0)$. Then*

$$X = M \uplus J_X^{-1} M^0 \text{ and } X^* = M^0 \uplus J_X M.$$

Further decomposition results will be given in the next section.

1.3 Representations of Compact Linear Operators

Let H be a Hilbert space and suppose that $A : H \to H$ is compact and self-adjoint. Then for all $x \in H$,

$$Ax = \sum_n \lambda_n (x, \phi_n) \phi_n,$$

where the λ_n are eigenvalues of A, each repeated according to multiplicity and ordered by decreasing modulus, while the ϕ_n are orthonormal eigenvectors of A corresponding to the eigenvalues λ_n; here $(\cdot, \cdot)$ denotes the inner product in H. Moreover, if the kernel of A is trivial, then the ϕ_n form a complete orthonormal set in H. All this is classical, as is its extension (by Erhard Schmidt) to arbitrary compact linear operators acting between (possibly different) Hilbert spaces; for details of the proofs see [41], Chap. II, Sect. 5, for example. It turns out (see [44]) that results of a broadly similar character to these can be obtained in a Banach space setting, under some restrictions on the spaces; we describe this work below and make applications in later chapters.

Throughout this section it will be supposed that X and Y are real, reflexive Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and that $T : X \to Y$ is compact and linear. The results to be given also hold for complex spaces, but with minor technical complications in the proofs that we prefer to avoid here. We begin with a simple result.

Proposition 1.10. *There exists $x_1 \in X$, with $\|x_1\|_X = 1$, such that $\|T\| = \|Tx_1\|_Y$.*

Proof. We may assume that $T \neq 0$ as otherwise the result is obvious. Let $\{w_k\}$ be a sequence of elements of X with $\|w_k\|_X = 1$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|Tw_k\|_Y = \|T\|$. Since $\{w_k\}$ is bounded and X is reflexive, there is a weakly convergent subsequence of $\{w_k\}$, still denoted by $\{w_k\}$ for convenience, with weak limit $w \in X$. As T is compact, $Tw_k \to Tw$. Thus $\|w\|_X \leq \liminf_{k\to\infty} \|w_k\|_X = 1$ and $\|Tw\|_Y = \|T\|$, from which it is immediate that $\|w\|_X = 1$. Now take $x_1 = w$. □

From now on we suppose additionally that X, Y, X^* and Y^* are strictly convex. These blanket assumptions, although not always necessary, allow us to streamline the presentation. By Proposition 1.3, given any $x \in X\backslash\{0\}$, there is a unique element of X^*, here written as $\widetilde{J}_X(x)$, such that $\left\|\widetilde{J}_X(x)\right\|_{X^*} = 1$ and $\left\langle x, \widetilde{J}_X(x)\right\rangle_X = \|x\|_X$; $\widetilde{J}_Y$ is defined in a similar way. For all $x \in X\backslash\{0\}$,

$$\widetilde{J}_X(x) = \text{grad } \|x\|_X,$$

where grad $\|x\|_X$ denotes the Gâteaux derivative of $\|\cdot\|_X$ at x. A corresponding relationship holds for $\widetilde{J}_Y$. Next, let μ_X, μ_Y be gauge functions that are normalised in the sense that $\mu_X(1) = \mu_Y(1) = 1$, and let J_X, J_Y be the corresponding duality maps in X, Y respectively. Then

$$J_X(x) = \mu\left(\|x\|_X\right)\widetilde{J}_X(x)\ (x \in X\backslash\{0\}), J_X(0) = 0,$$

the same holding if X is replaced by Y.

Proposition 1.11. *Let x_1 be as in Proposition 1.10 and suppose that $T \neq 0$. Then $x = x_1$ satisfies the equation*

$$T^*\widetilde{J}_Y T x = \nu \widetilde{J}_X x, \tag{1.22}$$

with $\nu = \|T\|$; in terms of duality maps this equation has the form

$$T^* J_Y T x = \nu_1 J_X x,\ \nu_1 = \|T\|\, \mu_Y(\|T\|). \tag{1.23}$$

Moreover, if $x \in X\backslash\{0\}$ satisfies (1.22) for some ν, then $0 \leq \nu \leq \|T\|$ and $\|Tx\|_Y = \nu\|x\|_X$.

Proof. Since

$$\|T\| = \|Tx_1\|_Y = \max_{x \in X\backslash\{0\}} \frac{\|Tx\|_Y}{\|x\|_X},$$

it follows that for all $x \in X$,

$$\frac{d}{dt}\left(\frac{\|Tx_1 + tTx\|_Y}{\|x_1 + tx\|_X}\right)\Bigg|_{t=0} = 0,$$

so that in terms of duality pairings,

$$\left\langle Tx, \widetilde{J}_Y Tx_1\right\rangle_Y = \|Tx_1\|_Y \left\langle x, \widetilde{J}_X x_1\right\rangle_X,$$

and hence

$$T^*\widetilde{J}_Y T x_1 = \lambda \widetilde{J}_X x_1,$$

with $\lambda = \|T\|$.

For the converse, let $x \in X\backslash\{0\}$ satisfy (1.22) for some ν. Then

$$\|Tx\|_Y = \left\langle Tx, \widetilde{J}_Y Tx \right\rangle_Y = \left\langle x, T^* \widetilde{J}_Y Tx \right\rangle_X = \nu \left\langle x, \widetilde{J}_X x \right\rangle_X = \nu \|x\|_X .$$

Hence $0 \le \nu \le \|T\|$. □

Proposition 1.11 shows that $\|T\|$ is the largest 'eigenvalue' ν of (1.22), which may be thought of as the Euler equation for maximising $\|Tx\|_Y$ subject to the condition $\|x\|_X = 1$. Note that when X and Y are Hilbert spaces, it reduces to

$$T^*Tx = \nu x,$$

so that $\nu^{1/2}$ is simply a singular value of T, in standard terminology. Note also that (1.23) may be written in the form

$$T^* J_Y T J_X^{-1} x^* = \nu_1 x^*,$$

so that ν_1 is an 'eigenvalue' of the (in general) nonlinear map $T^* J_Y T J_X^{-1}$, viewed as acting from X^* to X^*.

We now apply Propositions 1.10 and 1.11 repeatedly. Let $X = X_1$, put $M_1 =$ sp $\{J_X x_1\}$ (the linear span of $J_X x_1$), $X_2 = {}^0M_1$, $N_1 =$sp $\{J_Y T x_1\}$, $Y_2 = {}^0N_1$ and $\lambda_1 = \|T\|$. Both X_2 and Y_2 are reflexive as they are closed subspaces of reflexive spaces; also, X_2^* and Y_2^* are strictly convex, since, for example, $X_2^* = \left({}^0M_1\right)^*$ is isometrically isomorphic to X_1^*/M_1, which is strictly convex, by Proposition 1.9. Moreover, since by Proposition 1.11,

$$\langle Tx, J_Y T x_1 \rangle_Y = \nu_1 \langle x, J_X x_1 \rangle_X \text{ for all } x \in X,$$

it follows that T maps X_2 to Y_2. The restriction T_2 of T to X_2 is thus a compact linear map from X_2 to Y_2, and if it is not the zero operator, then by Proposition 1.11 there exists $x_2 \in X_2\backslash\{0\}$ such that, with obvious notation,

$$\langle T_2 x, J_{Y_2} T_2 x_2 \rangle_{Y_2} = \nu_2 \langle x, J_{X_2} x_2 \rangle_{X_2} \text{ for all } x \in X_2,$$

where $\nu_2 = \lambda_2 \mu_Y(\lambda_2), \lambda_2 = \|Tx_2\|_Y = \|T_2\|$. Evidently $\lambda_2 \le \lambda_1$. and $\nu_2 \le \nu_1$. Continuing in this way we obtain elements $x_1, x_2, ..., x_n$ of X, all with unit norm, subspaces $M_1, ..., M_n$ of X^* and $N_1, ..., N_n$ of Y^*, where

$$M_k = \text{ sp } \{J_X x_1, ..., J_X x_k\} \text{ and } N_k = \text{ sp } \{J_Y T x_1, ..., J_Y T x_k\},\ k = 1, ..., n,$$

and decreasing families $X_1, ..., X_n$ and $Y_1, ..., Y_n$ of subspaces of X and Y respectively given by

$$X_k = {}^0M_{k-1}, Y_k = {}^0N_{k-1},\ k = 2, ..., n. \tag{1.24}$$

Moreover, for each $k \in \{1,...,n\}$, T maps X_k into Y_k, $x_k \in X_k$ and with $T_k := T \restriction_{X_k}$, $\lambda_k = \|T_k\|$, $\nu_k = \lambda_k \mu_Y(\lambda_k)$, we have

$$\left\langle T_k x, J_{Y_k} T_k x_k \right\rangle_{Y_k} = \nu_k \left\langle x, J_{X_k} x_k \right\rangle_{X_k} \text{ for all } x \in X_k, \tag{1.25}$$

and so

$$T_k^* J_{Y_k} T_k x_k = \nu_k J_{X_k} x_k. \tag{1.26}$$

Note that (1.25) is equivalent to

$$\langle T_k x, J_Y T x_k \rangle_Y = \nu_k \langle x, J_X x_k \rangle_X \text{ for all } x \in X_k. \tag{1.27}$$

For on identifying Y_k^* with the quotient space Y^*/Y_k^0, it follows that $J_{Y_k} y - J_Y y \in Y_k^0$ for any $y \in Y_k$ and hence, if $x \in X_k$,

$$\left\langle T_k x, J_{Y_k} y \right\rangle_{Y_k} = \langle T_k x, J_Y y \rangle_Y$$

since $T_k x \in Y_k$. The right-hand sides of (1.25) and (1.27) are handled in a similar way.

Since $T x_k \in Y_k = {}^0 N_{k-1}$,

$$\langle T x_k, J_Y T x_l \rangle_Y = 0 \text{ if } l < k. \tag{1.28}$$

In terms of the semi-inner product $(\cdot,\cdot)_X$ defined by (1.18), this means that

$$(x_l, x_k)_X = 0 \text{ if } l < k. \tag{1.29}$$

The process stops with λ_n, x_n and X_{n+1} if and only if the restriction of T to X_{n+1} is the zero operator. In that case, the range of T is the linear space spanned by $Tx_1,...,Tx_n$. For if $x \in X$, put

$$w_k = x - \sum_{j=1}^{k-1} \xi_j x_j, \ \xi_j = \xi_j(x),$$

for $k \geq 2$, where the ξ_j are so chosen that $w_k \in X_k$. Such a choice is possible, and in a unique way, in view of (1.29): just take $\xi_1 = \langle x, J_X x_1 \rangle_X$ and for $2 \leq l \leq k-1$,

$$\xi_l = \left\langle x - \sum_{j=1}^{l-1} \xi_j x_j, J_X x_l \right\rangle_X.$$

Hence $T w_{n+1} = 0$, so

$$Tx = \sum_{j=1}^{n} \xi_j T x_j = \sum_{j=1}^{n} \lambda_j \xi_j y_j, \text{ where } y_j = T x_j / \left\|T x_j\right\|_Y. \tag{1.30}$$

This situation may occur even if X is infinite-dimensional; it will certainly be the case eventually if $\dim X < \infty$ since $x_1,\ldots,x_n$ are linearly independent. We observe that as the ξ_k above satisfy

$$\xi_k = (x_k,x)_X - \sum_{j=1}^{k-1} \xi_j (x_k,x_j)_X, k = 2,\ldots,n, \tag{1.31}$$

then with $\widehat{\xi} := (\xi_1,\ldots,\xi_n)^t$, where the superscript t denotes the transpose, we have the matrix equation

$$\Gamma(x_1,\ldots,x_n)\widehat{\xi} = ((x_1,x)_X,\ldots,(x_n,x)_X)^t, \tag{1.32}$$

where $\Gamma(x_1,\ldots,x_n)$ is the lower-triangular, Gram-type $n \times n$ matrix with kth row

$$(x_k,x_1)_X,\ldots,(x_k,x_{k-1})_X, 1,0,\ldots,0$$

if $k \geq 2$, and first row $1,0,\ldots,0$. Since $\Gamma(x_1,\ldots,x_n)$ is invertible, we have an explicit formula for the ξ_k.

Proposition 1.12. *If T is not of finite rank, then the sequence $\{\lambda_n\}$ is infinite and converges to zero.*

Proof. Since $Tx_n \in {}^0N_{n-1}$,

$$\left\langle Tx_n, \widetilde{J_Y} Tx_m \right\rangle_Y = 0 \text{ if } m < n. \tag{1.33}$$

Thus if $m < n$,

$$\begin{aligned}\lim_{k\to\infty} \lambda_k \leq \|Tx_m\|_Y &= \left\langle Tx_m, \widetilde{J_Y} Tx_m \right\rangle_Y = \left\langle Tx_m - Tx_n, \widetilde{J_Y} Tx_m \right\rangle_Y \\ &\leq \|Tx_m - Tx_n\|_Y \left\| \widetilde{J_Y} Tx_m \right\|_{Y^*} = \|Tx_m - Tx_n\|_Y .\end{aligned}$$

Since $\{x_n\}$ is bounded and T is compact, some subsequence of $\{Tx_n\}$ must converge. The result follows. □

There is an obvious connection between the λ_n and the Gelfand numbers $c_n(T)$ of T. These numbers form one of several sequences used to help determine 'how compact T is' and are defined by

$$c_n(T) = \inf \left\| T \restriction_{\widetilde{X_n}} \right\| \ (n \in \mathbb{N}),$$

where the infimum is taken over all linear subspaces $\widetilde{X_n}$ of X with codim $\widetilde{X_n} < n$ (see [41], Definition II.3.3). Together with other related numbers they will be discussed more fully later on. Since

$$\operatorname{codim} X_k = \dim(\operatorname{sp}\{J_X x_1, ..., J_X x_{k-1}\}),$$

it follows immediately that

$$c_n(T) \leq \lambda_n \ (n \in \mathbb{N}).$$

Corollary 1.2. *If the rank of T is infinite, then*

$$\ker T \supset \cap_{n\in\mathbb{N}} X_n := X_\infty. \tag{1.34}$$

Proof. If $x \in \cap_{n\in\mathbb{N}} X_n$, then for all $n \in \mathbb{N}$, $\|Tx\| \leq \lambda_n \|x\| \to 0$ as $n \to \infty$: thus $x \in \ker T$. □

Next we examine the properties of the family of maps

$$S_k : X \to \mathscr{M}'_{k-1} := \operatorname{sp}\{x_1, ..., x_{k-1}\} \ (k \geq 2)$$

determined by the condition that $x - S_k x \in X_k$ for all $x \in X$. By induction it follows that S_k is uniquely given by

$$S_k x := \sum_{j=1}^{k-1} \xi_j(x) x_j, \tag{1.35}$$

where, as noted above,

$$\xi_j(x) = \left\langle x - \sum_{i=1}^{j-1} \xi_i(x) x_i, J_X x_j \right\rangle_X, \ \xi_1(x) = \langle x, J_X x_1 \rangle_X .$$

Hence S_k is linear. From the uniqueness it follows that $S_k^2 = S_k$ and S_k is a linear projection of X onto $\mathscr{M}'_{k-1}$. With $E_j := \langle \cdot, J_X x_j \rangle_X x_j$ we have

$$S_k x = S_{k-1} x + \langle x - S_{k-1} x, J_X x_{k-1} \rangle_X x_{k-1}$$

and so

$$S_k = S_{k-1} + E_{k-1}(I - S_{k-1}),$$

where I is the identity map of X onto itself. Hence

$$I - S_k = (I - E_{k-1}) \cdots (I - E_1) \ (k \in \mathbb{N}, k \geq 2). \tag{1.36}$$

Proposition 1.13. *The spaces X and X^* have the direct sum decompositions*

$$X = X_k \oplus \mathscr{M}'_{k-1}, \ X^* = M_{k-1} \oplus \left(\mathscr{M}'_{k-1}\right)^0 \tag{1.37}$$

for each $k \geq 2$. *The operators* S_k, S_k^* *are respectively linear projections of* X *onto* $\mathscr{M}'_{k-1}$ *and* X^* *onto* M_{k-1}.

Proof. From (1.36) we have

$$I^* - S_k^* = (I^* - E_1^*) \cdots (I^* - E_{k-1}^*), \tag{1.38}$$

and it is readily seen that

$$E_j^* = \langle x_j, \cdot \rangle_X J_X x_j.$$

It easily follows by induction that S_k^* and $I^* - S_k^*$ have ranges M_{k-1} and $\left(\mathscr{M}'_{k-1}\right)^0$ respectively. □

Remark 1.4. In view of the above Proposition, we can write the identity (1.27) as

$$\langle T(I - S_k)x, J_Y T x_k \rangle_Y = \nu_k \langle (I - S_k)x, J_X x_k \rangle_X \text{ for all } x \in X. \tag{1.39}$$

Hence

$$(T^* J_Y T - \nu_k J_X) x_k \in X_k^0 = M_{k-1}$$

and

$$(I^* - S_k^*)(T^* J_Y T - \nu_k J_X) x_k = 0. \tag{1.40}$$

We summarise some of the previous results in the following theorem.

Theorem 1.7. *Suppose that* T *is not of finite rank. Then for each* $k \in \mathbb{N}$, *there exist* $x_k \in X_k$ *and* $\nu_k \in \mathbb{R}$ *such that (1.26), (1.27) and (1.40) (*$k \geq 2$ *for this last equation) are satisfied with* $\lim_{k\to\infty} \nu_k = 0$.

Remark 1.5. Suppose that $X = Y$. If λ is a non-zero eigenvalue of T corresponding to a normalised eigenvector x, then

$$\left\langle x, \lambda^{-1} T^* \widetilde{J}_X x \right\rangle_X = \left\langle \lambda^{-1} T x, \widetilde{J}_X x \right\rangle_X = \left\langle x, \widetilde{J}_X x \right\rangle_X = 1,$$

so that by the strict convexity and reflexivity of X^* (see Proposition 1.3),

$$T^* \widetilde{J}_X x = \lambda \widetilde{J}_X x.$$

This means that $\widetilde{J}_X x$ is an eigenvector of T^* with corresponding eigenvalue λ; so is $J_X x$. Moreover, since $\widetilde{J}_X(\lambda x) = (\text{sgn } \lambda)\widetilde{J}_X x$,

$$T^* \widetilde{J}_X T x = T^* \widetilde{J}_X(\lambda x) = (\text{sgn } \lambda) T^* \widetilde{J}_X x = |\lambda| \widetilde{J}_X x,$$

and hence the eigenvector x of T satisfies (1.22). Consideration of suitable compact Volterra integral operators shows that solutions of (1.22) need not be eigenvectors of T.

When X and Y are Hilbert spaces, the duality maps are identified with the identity and the dual spaces with the original spaces; also $S_k^* = S_k$. The direct sums in Proposition 1.13 are now orthogonal sums and, by (1.28), $T^*Tx_k \in X_k$. Hence $S_k(T^*T - \nu_k I)x_k = 0$. Use of (1.40) now leads to $|T|^2 x_k := T^*Tx_k = \lambda_k^2 x_k$, since $\nu_k = \lambda_k^2$. Hence λ_k is a singular value of T. In this case, our result gives the classical Schmidt decomposition of T and T^* (see, for example, Chap. II, Sect. 5 of [41]). This is just what is needed in the general, non-Hilbertian, case that we now discuss, in which we continue to assume that X and Y are real and reflexive, with strictly convex duals.

Given $x \in X$ and $k \in \mathbb{N}$, let z_k be the point in X_k nearest to x, so that $z_k = P_k x$, where P_k is the projection of X onto X_k. As $\|z_k - x\| \leq \|x\|$, it follows that $\|z_k\| \leq 2\|x\|$; and of course $\|Tz_k\| \leq \lambda_k \|z_k\|$. Since

$$\|z_k - x\| = \inf\{\|x - z_k + ty\| : y \in X_k\},$$

we see that for all $y \in X_k$,

$$\frac{d}{dt}\|x - z_k + ty\| \restriction_{t=0} = 0,$$

and hence

$$\langle y, J_X(x - z_k)\rangle = 0 \text{ for all } y \in X_k.$$

It follows that $J_X(x - z_k) \in M_{k-1}$. Hence, by Proposition 1.13,

$$J_X(x - P_k x) = S_k^* J_X(x - P_k x),$$

and so

$$x - P_k x = J_X^{-1} S_k^* J_X(x - P_k x).$$

Since $\lambda_k \to 0$ as $k \to \infty$,

$$Tx = \lim_{k\to\infty} T J_X^{-1} S_k^* J_X(x - P_k x).$$

From Corollary 1.2 it follows that as T has infinite rank, there is a strictly increasing sequence $(k(j))_{j\in\mathbb{N}}$ of natural numbers such that the weak limit $w - \lim_{j\to\infty} z_{k(j)}$ exists and lies in ker (T). If ker $(T) = \{0\}$, then $z_{k(j0} \to 0$ weakly and hence we have the following Proposition.

Proposition 1.14. *Suppose that T has infinite rank and trivial kernel. Then for all $x \in X$,*

$$x = w - \lim_{j\to\infty} J_X^{-1} S_{k(j)}^* J_X(x - P_{k(j)} x).$$

We can write this expression in the form

$$x = w - \lim_{j\to\infty} J_X^{-1}\left(\sum_{i=1}^{k(j)-1} \eta_i(k(j), x) J_X x_i\right)$$

for some real constants $\eta_i(k(j),x)$. We shall show below that the convergence is strong if X is additionally assumed to be uniformly convex. Since each η_i depends on $k(j)$ and x, and J_X is (in general) non-linear, this does not imply that the $J_X x_i$ form a basis of X. However, if X is a Hilbert space, so that J_X is identified with the identity map of X to itself, and ker $(T) = \{0\}$, then the fact that x is the weak limit of some linear combinations of the x_i implies that it is also the strong limit of some linear combination of the x_i, and we may write

$$x = \sum_{i=1}^{\infty} \widetilde{\eta}_i x_i$$

in the sense of strong convergence, where now $\widetilde{\eta} = (x, x_i)_X$. Then

$$Tx = \sum_{i=1}^{\infty} \widetilde{\eta}_i T x_i = \sum_{i=1}^{\infty} \lambda_i \widetilde{\eta}_i y_i,\ y_i = Tx_i / \|Tx_i\|.$$

We summarise the last results as follows.

Theorem 1.8. *Suppose that X is a Hilbert space and that T has infinite rank and trivial kernel. Then the x_i form a basis of X, so that any $x \in X$ may be represented in the form*

$$x = \sum_{i=1}^{\infty} \widetilde{\eta}_i x_i,\ \widetilde{\eta} = (x, x_i)_X.$$

The action of T is then described by

$$Tx = \sum_{i=1}^{\infty} \lambda_i \widetilde{\eta}_i y_i,\ y_i = Tx_i / \|Tx_i\|.$$

While this result gives a reasonably satisfying description of the position when the domain space X is a Hilbert space, the situation without this restriction is less agreeable: for example, the formula in Proposition 1.14 involves weak convergence. To show that this can be replaced by strong convergence, we need the following lemma, in which to simplify notation we have omitted the canonical maps of X onto X/L and X/S. We continue with this abuse of notation whenever the likelihood of ambiguity is remote, including canonical maps if clarification seems desirable.

Lemma 1.2. *Let $\mathscr{L}(X)$ be the set of all closed linear subspaces of X, let $\mathscr{S} \subset \mathscr{L}(X)$ and put $L = \cap_{S \in \mathscr{S}} S$, $N = \overline{\cup_{S \in \mathscr{S}} S^0}$. Then $L^0 = N$ and for all $x \in X$,*

$$\|x\|_{X/L} = \sup_{S \in \mathscr{S}} \|x\|_{X/S}. \tag{1.41}$$

Proof. Let $\Phi : \mathscr{L}(X) \to \mathscr{L}(X^*)$ be given by $\Phi(Z) = Z^0$ $(Z \in \mathscr{L}(X))$. We observe that given any linear subspace G of a Banach space, its polar G^0 is closed and

$\overline{G} = {}^0(G^0)$; a corresponding result holds for subspaces of the dual space. Hence Φ is bijective; it also reverses inclusion. Thus for any $S \in \mathscr{S}$, $L \subset S$ and so $S^0 \subset L^0$; hence $\cup_{S\in\mathscr{S}} S^0 \subset L^0$ and $N \subset L^0$. Moreover, for all $S \in \mathscr{S}$, $S^0 \subset N$, which shows that $\Phi^{-1}(N) \subset \cap_{S\in\mathscr{S}} S = L$, whence $L^0 \subset N$. Consequently $L^0 = N$. This implies that $(X/L)^*$ is isometrically isomorphic to N, and

$$\begin{aligned} \sup\{|\langle x,y\rangle_X| : y \in N, \|y\|_{X^*} \le 1\} &= \sup\{|\langle x,y\rangle_X| : y \in \cup_{S\in\mathscr{S}} S^0, \|y\|_{X^*} \le 1\} \\ &= \sup_{S\in\mathscr{S}} \left\{ \sup\{|\langle x,y\rangle_X| : y \in S^0, \|y\|_{X^*} \le 1\} \right\}. \end{aligned}$$

The result follows. □

Lemma 1.3. *Suppose that X is uniformly convex, let $X_\infty = \cap_{k\in\mathbb{N}} X_k$ and write P_k, P_∞ for the projections P_{X_k}, P_{X_∞} introduced earlier. Then for all $x \in X, P_k x \to P_\infty x$ as $k \to \infty$.*

Proof. Since $\|x - P_k x\| = \|x\|_{X/X_k} \le \|x\|_X$, it follows that $\|P_k x\| \le 2\|x\|$; hence $\{P_k x\}$ has a subsequence that converges weakly to $y \in X_\infty$, say. We claim that $y = P_\infty x$. For if not, then

$$\|x - y\| > \|x - P_k x\| = \|x\|_{X/X_\infty}.$$

Thus

$$\begin{aligned} \|x - P_k x\| &\ge \left\langle x - P_k x, \widetilde{J}_X(x-y) \right\rangle \to \left\langle x - y, \widetilde{J}_X(x-y) \right\rangle \\ &= \|x - y\| > \|x\|_{X/X_\infty}, \end{aligned}$$

and this implies that for some $k \in \mathbb{N}$, $\|x - P_k x\| > \|x\|_{X/X_\infty}$. This means that $\|x\|_{X/X_k} > \|x\|_{X/X_\infty}$, which contradicts the fact that $X_\infty \subset X_k$. Thus every weakly convergent subsequence of $\{P_k x\}$ has weak limit $P_\infty x$, from which it follows by a standard contradiction argument that the whole sequence $\{P_k x\}$ converges weakly to $P_\infty x$. By Lemma 1.2, $\|x - P_k x\| \to \|x - P_\infty x\|$. The result now follows from the uniform convexity of X.

Theorem 1.9. *Let X be uniformly convex. Then for all $x \in X$,*

$$x = \lim_{k\to\infty} (I - P_k) S_k x + P_\infty x.$$

If ker $(T) = \{0\}$ and $\lim_{k\to\infty} S_k x$ exists, then $x = \sum_{j=1}^\infty \xi_j(x) x_j$.

Proof. For any closed linear subspace L of X and any $u \in X$, $P_L u$ is the unique element $w \in L$ for which $\|u - w\|$ is minimal. Hence if $u - v \in L$ we have $P_L(u - v) = u - v$, and since

$$\|u - (u - v + P_L v)\| = \|v - P_L v\|,$$

we also have $P_L u = u - v + P_L v$, so that

$$P_L u - P_L v = u - v = P_L(u - v).$$

With $s_k = s_k(x) := S_k x$,

$$(s_k - P_k s_k) - (x - P_k x) = s_k - x - P_k(s_k - x) = 0.$$

As $x - P_k x \to x - P_\infty x$, by Lemma 1.3, the proof of the expression for x is complete. The rest is immediate, for if ker $(T) = \{0\}$ and $\lim_{k\to\infty} s_k$ exists, then $\lim_{k\to\infty} P_k s_k$ exists and is 0, since $X_\infty = \{0\}$.

Note that if X is a Hilbert space, then $S_k x \in \mathscr{M}'_{k-1} = X_k^\perp$ and hence $P_k S_k x = 0$. We thus have

$$x = \sum_{j=1}^{\infty} (x, x_j)_X x_j + P_\infty x.$$

Results corresponding to those given above can be obtained if the hypothesis of uniform convexity is made about Y rather than X. Thus suppose that Y is uniformly convex, put $Y_\infty = \cap_{k=1}^{\infty} Y_k$ and let Q_k, Q_∞ be the projections of Y onto Y_k, Y_∞ respectively. Then procedures similar to those given above show that

$$Tx = \lim_{k\to\infty} (I - Q_k) T S_k x + Q_\infty T x,$$

where $TS_k x = \sum_{j=1}^{k-1} \xi_j(x) T x_j = \sum_{j=1}^{k-1} \lambda_j \xi_j(x) y_j$. If Y is a Hilbert space, then for all $x \in X$,

$$Tx = \sum_{j=1}^{\infty} \lambda_j \xi_j(x) y_j + Q_\infty T x.$$

Further details may be found in [44].

Notes

Note 1.1. For further information about the classical theory of bases, see [73] and [114]. Among the more interesting recent developments concerning unconditional bases are those involving the notion of indecomposability: an infinite-dimensional Banach space X is called indecomposable if the only representations $X = Y \bigoplus Z$, where Y and Z are closed subspaces of X, are those in which either Y or Z is finite-dimensional; X is hereditarily indecomposable (HI) if every infinite-dimensional closed subspace of X is indecomposable. An example of an HI space is given in [69]. The Gowers dichotomy theorem (see [68]) asserts that if X is a Banach space, then there is a subspace Y of X such that either Y is HI or Y has an unconditional basis.

Note 1.2. The property of uniform convexity behaves well under interpolation. Let A_0, A_1 be a compatible pair of Banach spaces, one of which is uniformly convex, and let $\theta \in (0,1)$, $p \in (1,\infty)$; denote by $(A_0, A_1)_{\theta,p}$, $[A_0, A_1]_\theta$ the spaces obtained from A_0 and A_1 by real and complex interpolation, respectively. Then $(A_0,A_1)_{\theta,p}$ is uniformly convex (see [4], p.71), and so is $[A_0,A_1]_\theta$ (see [27]): note, however, that the norm used in [4] is not the familiar K-functional norm, although it is equivalent to that norm. As the Lebesgue spaces $L_r(\Omega)$ (where Ω is, for example, a measurable subset of $\mathbb{R}^n$) are uniformly convex if $r \in (1,\infty)$, it follows that the Lorentz spaces $L_{p,q}(\Omega)$ are also uniformly convex if $p,q \in (1,\infty)$. For by [120], Theorem 1.18.6/2, p. 134,

$$L_{p,q}(\Omega) = \left(L_{p_0}(\Omega), L_{p_1}(\Omega)\right)_{\theta,q},$$

where p_0, p_1 and θ are so chosen that $p_0, p_1 \in (1,\infty)$, $p_0 \neq p_1$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. We refer to the papers by Kamińska [77–79] for the uniform convexity of Orlicz and Orlicz–Lorentz spaces. From her work follows the uniform convexity of $L_{p,q}(\Omega)$ endowed with the standard Lorentz norm.

For background material and further results concerning duality maps, see [19,24, 40,96]. Details of how duality maps on a space X naturally induce duality maps on quotient spaces of X are given in [44].

Note 1.3. Various attempts have been made to extend to Banach spaces the classical theory of the representation of compact linear operators acting between Hilbert spaces. Much of this work is based on quite severe restrictions on the class of compact operators considered: see, for example, [118] and [23]. The approach of [44] that we present here is different: general compact operators are considered, but the spaces are typically restricted to being reflexive and strictly convex, with strictly convex duals. For an account of work of this type emphasising the rôle of James orthogonality, see [47].

Chapter 2
Trigonometric Generalisations

In this chapter we introduce the p-trigonometric functions, for $1 < p < \infty$, and establish their fundamental properties. These functions generalise the familiar trigonometric functions, coincide with them when $p = 2$, and otherwise have important similarities to and differences from their classical counterparts. As will be shown later, they play an important part in both the theory of the p-Laplacian and that of the Hardy operator. Particular attention is paid to the basis properties of the analogues of the sine functions in the context of Lebesgue spaces.

2.1 The Functions $\sin_p$ and $\cos_p$

Let $1 < p < \infty$ and define a (differentiable) function $F_p : [0,1] \to \mathbb{R}$ by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt. \tag{2.1}$$

Plainly $F_2 = \arcsin$. Since F_p is strictly increasing it has an inverse which, by analogy with the case $p = 2$, we denote by $\sin_p$. This is defined on the interval $[0, \pi_p/2]$, where

$$\pi_p = 2\int_0^1 (1-t^p)^{-1/p} dt. \tag{2.2}$$

Thus $\sin_p$ is strictly increasing on $[0, \pi_p/2]$, $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$. We extend $\sin_p$ to $[0, \pi_p]$ by defining

$$\sin_p(x) = \sin_p(\pi_p - x) \text{ for } x \in [\pi_p/2, \pi_p]; \tag{2.3}$$

further extension to $[-\pi_p, \pi_p]$ is made by oddness; and finally $\sin_p$ is extended to the whole of $\mathbb{R}$ by $2\pi_p$-periodicity. It is clear that this extension is continuously differentiable on $\mathbb{R}$.

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A function $\cos_p : \mathbb{R} \to \mathbb{R}$ is defined by the prescription

$$\cos_p(x) = \frac{d}{dx}\sin_p(x),\ x \in \mathbb{R}. \tag{2.4}$$

Evidently $\cos_p$ is even, $2\pi_p$-periodic and odd about $\pi_p/2$. If $x \in [0, \pi_p/2]$ and we put $y = \sin_p(x)$, then

$$\cos_p(x) = (1 - y^p)^{1/p} = (1 - (\sin_p(x))^p)^{1/p}. \tag{2.5}$$

Thus $\cos_p$ is strictly decreasing on $[0, \pi_p/2]$, $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. Also

$$\left|\sin_p x\right|^p + \left|\cos_p x\right|^p = 1; \tag{2.6}$$

this is immediate if $x \in [0, \pi_p/2]$, but it holds for all $x \in \mathbb{R}$ in view of symmetry and periodicity. Note that the analogy between these p-functions and the classical trigonometric functions is not complete. For example, while the extended $\sin_p$ function belongs to $C^1(\mathbb{R})$, it is far from being real analytic on $\mathbb{R}$ if $p \neq 2$. To see this, observe that with the aid of (2.6) its second derivative at $x \in [0, \pi_p/2)$ can be shown to be $-h(\sin_p x)$, where

$$h(y) = (1 - y^p)^{\frac{2}{p}-1} y^{p-1},$$

and so is not continuous at $\pi_p/2$ if $2 < p < \infty$. Nevertheless, $\sin_p$ is real analytic on $[0, \pi_p/2)$. Figure 2.1 below gives the graphs of $\sin_p$ and $\cos_p$ for $p = 1.2$ and $p = 6$.

To calculate π_p we make the change of variable $t = s^{1/p}$ in the formula above for π_p. Then

$$\pi_p/2 = p^{-1}\int_0^1 (1-s)^{-1/p} s^{1/p-1} ds = p^{-1} B(1-1/p, 1/p) = p^{-1}\Gamma(1-1/p)\Gamma(1/p),$$

where B is the Beta function. Hence

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)}. \tag{2.7}$$

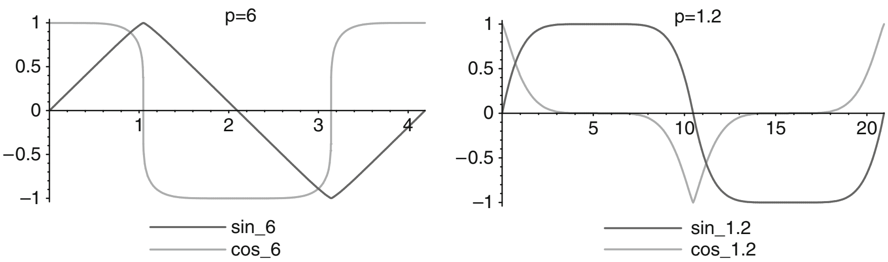

Fig. 2.1 $\sin_6, \cos_6$ and $\sin_{1.2}, \cos_{1.2}$

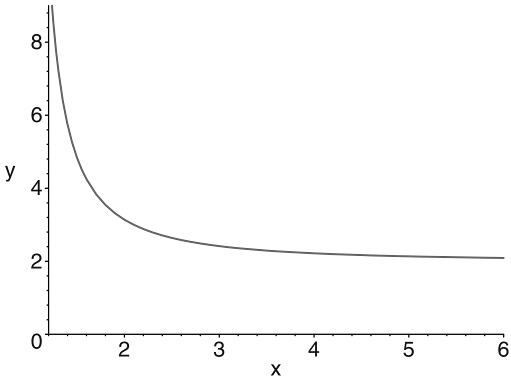

Fig. 2.2 $y = \pi_p$

Note that $\pi_2 = \pi$ and

$$p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \tag{2.8}$$

Using (2.7) and (2.8) we see that π_p decreases as p increases, with

$$\lim_{p\to 1} \pi_p = \infty, \ \lim_{p\to\infty} \pi_p = 2, \ \lim_{p\to 1}(p-1)\pi_p = \lim_{p\to 1} \pi_{p'} = 2. \tag{2.9}$$

The dependence of π_p on p is illustrated by Fig. 2.2.

An analogue of the tangent function is obtained by defining

$$\tan_p x = \frac{\sin_p x}{\cos_p x} \tag{2.10}$$

for those values of x at which $\cos_p x \neq 0$. This means that $\tan_p x$ is defined for all $x \in \mathbb{R}$ except for the points $(k+1/2)\pi_p$ $(k \in \mathbb{Z})$. Plainly $\tan_p$ is odd and π_p-periodic; also $\tan_p 0 = 0$. Some idea of the dependence of $\tan_p$ on p is provided by Fig. 2.3, in which the graph of this function is given for $p = 1.2$ and $p = 6$.

Use of (2.6) shows that on $(-\pi_p/2, \pi_p/2)$, $\tan_p$ has derivative $1 + \left|\tan_p x\right|^p$; and so if the inverse of $\tan_p$ on this interval is denoted by A, it follows that

$$A'(t) = 1/(1+|t|^p), \ t \in \mathbb{R}.$$

When $p = 2$, $A(t)$ is simply arctan t, giving a direct connection with an angle. To provide a similar geometric interpretation when $p \neq 2$ we follow Elbert [57] and endow the plane $\mathbb{R}^2$ with the l_p metric, so that the distance between points (x_1, x_2) and (y_1, y_2) of $\mathbb{R}^2$ is

$$\{|x_1 - y_1|^p + |x_2 - y_2|^p\}^{1/p}.$$

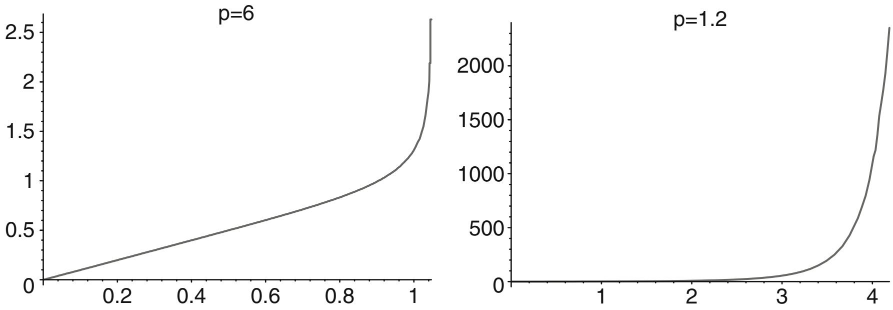

Fig. 2.3 $y = \tan_6(x),\ [0, \pi_6/2)$ $y = \tan_{1.2}(x),\ [0, \pi_{1.2}/2)$

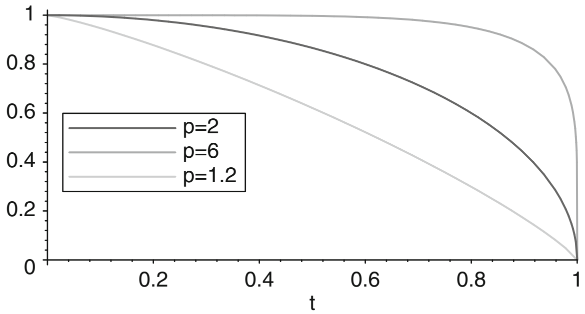

Fig. 2.4 The first quadrant of S_1 for $p = 2, 6, 1.2$

Given $R > 0$, when $1 < p < \infty$ the curve in $\mathbb{R}^2$ defined by $|x|^p + |y|^p = R^p$ will be called the p-circle with radius R, or the unit p-circle S_p if $R = 1$. The first quadrant of S_p is illustrated for $p = 1.2, 2, 4$ in Fig. 2.4.

Since $\left|\sin_p t\right|^p + \left|\cos_p t\right|^p = 1$, the p-circle of radius R may be parametrised by

$$x = R\cos_p t,\ y = \ R\sin_p t\ (0 \le t \le 2\pi_p), \tag{2.11}$$

just as in the familiar case in which $p = 2$. Let $P_1 = (\cos_p t, \sin_p t) \in I_p$ for some $t \in (0, 2\pi_p)$; we shall refer to t as the angle between the ray OP_1 (where $O = (0,0)$) and the positive x_1-axis. Now put

$$C_p(t) = \int_t^{\pi_p/2} \sin_p s\, ds$$

and let C be the curve $\left\{(C_p(t), \sin_p t) : t \in [0, 2\pi_p]\right\}$. The arc length of that part of C between $P_0 = (C_p(0), 0)$ and $P_2 = (C_p(t), \sin_p t)$, measured by means of the l_p metric on $\mathbb{R}^2$, is

$$\int_0^t \left\{\left|C_p'(s)\right|^p + \left|\cos_p s\right|^p\right\}^{1/p} ds = \int_0^t \left\{\left|\sin_p s\right|^p + \left|\cos_p s\right|^p\right\}^{1/p} ds = t.$$

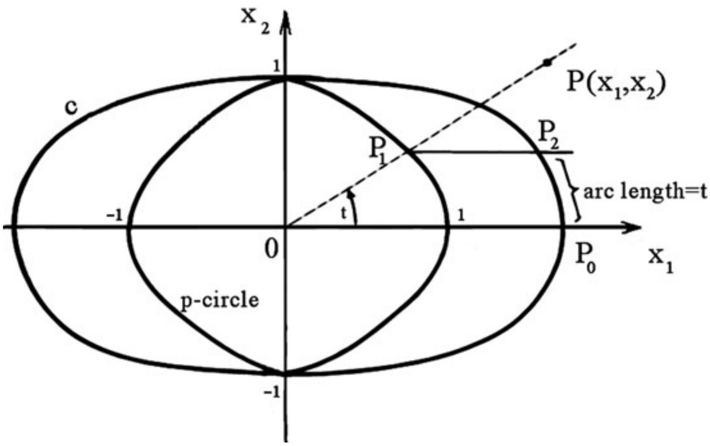

Fig. 2.5 Angles

This enables us to explain our method of measuring angles as follows. The ray OP (where $P=(x_1,x_2)$) meets the unit p-circle at $P_1=(\cos_p t,\sin_p t)$; the line through P_1 parallel to the x_1-axis meets C in the same quadrant of the plane at P_2 : see Fig. 2.5 (based on [57]).

Then the signed length of the arc P_0P_2, namely t, is our measure of the angle $P_0\widehat{O}P$: such a procedure corresponds to what is done when $p=2$. Note also that $x_2/x_1=\sin_p t/\cos_p t=\tan_p t$, so that the arc length $t=A(x_2/x_1)$. This enables us to introduce polar coordinates ρ and θ in $\mathbb{R}^2$ by

$$\rho=\left(|x_1|^p+|x_2|^p\right)^{1/p},\ \theta=A(x_2/x_1).$$

Next we record some basic facts about derivatives of the p-trigonometric functions. They follow immediately from the definitions and (2.6).

Proposition 2.1. *For all* $x\in[0,\pi_p/2)$,

$$\frac{d}{dx}\cos_p x=-\sin_p^{p-1}x\cos_p^{2-p}x,\ \frac{d}{dx}\tan_p x=1+\tan_p^p x,$$

$$\frac{d}{dx}\cos_p^{p-1}x=-(p-1)\sin_p^{p-1}x,\ \frac{d}{dx}\sin_p^{p-1}x=(p-1)\sin_p^{p-2}x\cos_p x.$$

Some elementary identities are provided in the proposition below.

Proposition 2.2. *For all* $y\in[0,1]$,

$$\cos_p^{-1}y=\sin_p^{-1}(1-y^p)^{1/p},\ \sin_p^{-1}y=\cos_p^{-1}(1-y^p)^{1/p}$$

and

$$\frac{2}{\pi_p}\sin_p^{-1} y^{1/p} + \frac{2}{\pi_{p'}}\sin_{p'}^{-1}(1-y^p)^{1/p'} = 1,\ \cos_p^p(\pi_p y/2) = \sin_{p'}^{p'}(\pi_{p'}(1-y)/2).$$

Proof. The first two claims follow directly from (2.6). For the third, note that

$$\sin_{p'}^{-1}(1-y^p)^{1/p'} = \int_0^{(1-y^p)^{1/p'}} (1-t^{p'})^{-1/p'}dt,$$

and that the change of variable $s = (1-t^{p'})^{1/p}$ transforms this integral into

$$\frac{p}{p'}\int_y^1 (1-s^p)^{-1/p}ds = \frac{p}{p'}\left(\frac{\pi_p}{2} - \sin_p^{-1} y\right) = \frac{\pi_{p'}}{\pi_p}\left(\frac{\pi_p}{2} - \sin_p^{-1} y\right),$$

the final step following from (2.8). To obtain the fourth identity, write

$$\cos_p^p(\pi_p y/2) = 1 - \sin_p^p(\pi_p y/2) := 1-x$$

and observe that in view of the third identity,

$$y = \frac{2}{\pi_p}\sin_p^{-1} x^{1/p} = 1 - \frac{2}{\pi_{p'}}\sin_{p'}^{-1}(1-x)^{1/p'},$$

which gives

$$1-x = \sin_{p'}^{p'}(\pi_{p'}(1-y)/2). \qquad \square$$

It is also convenient to have more refined extensions of the trigonometric functions. To obtain these, suppose first that $p,q \in (1,\infty)$ and put

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p}dt. \tag{2.12}$$

This coincides with π_p when $p = q$. Use of the substitution $s = t^q$ shows that

$$\pi_{p,q} = 2q^{-1}\int_0^1 (1-s)^{-1/p}s^{1/q-1}ds = 2q^{-1}B(1/p', 1/q). \tag{2.13}$$

From (2.12) it is easy to see that $\pi_{p,q}$ decreases as either p or q increases, the other being held constant, and that

$$\lim_{p\to\infty}\pi_{p,q} = 2\ (1<q<\infty), \lim_{q\to\infty}\pi_{p,q} = 2\ (1<p<\infty). \tag{2.14}$$

By analogy with the case $p = q$ we define $\sin_{p,q}$ on the interval $[0, \pi_{p,q}/2]$ to be the inverse of the strictly increasing function $F_{p,q} : [0,1] \to [0, \pi_{p,q}/2]$ given by

$$F_{p,q}(x) = \int_0^x (1-t^q)^{-1/p} dt. \tag{2.15}$$

This is then extended to all of the real line by the same processes involving symmetry and $2\pi_{p,q}$-periodicity as for the case $p = q$. The function $\cos_{p,q}$ is defined to be the derivative of $\sin_{p,q}$, and it follows easily that for all $x \in \mathbb{R}$,

$$\left|\sin_{p,q} x\right|^q + \left|\cos_{p,q} x\right|^p = 1. \tag{2.16}$$

So far we have supposed that $p, q \in (1, \infty)$, but with natural interpretations of the integrals involved the extreme values 1 and ∞ can be allowed. This gives

$$\pi_{p,q} = \begin{cases} 2p', & \text{if } 1 \le p \le \infty, q = 1, \\ 2, & \text{if } 1 \le p \le \infty, q = \infty, \\ \infty, & \text{if } p = 1, 1 \le q < \infty, \\ 2, & \text{if } p = \infty, 1 \le q \le \infty. \end{cases} \tag{2.17}$$

Corresponding values of $\sin_{p,q}$ and $\cos_{p,q}$ are given by

$$\sin_{p,q} x = \begin{cases} 1-(1-x/p')^{p'}, & \text{if } 1 < p \le \infty, q = 1, \\ x, & \text{if } 1 \le p \le \infty, q = \infty, \\ x, & \text{if } p = \infty, 1 \le q \le \infty, \end{cases} \tag{2.18}$$

and

$$\cos_{p,q} x = \begin{cases} (1-x/p')^{1/(p-1)}, & \text{if } 1 < p \le \infty, q = 1, \\ 1, & \text{if } 1 \le p \le \infty, q = \infty, \\ 1, & \text{if } p = \infty, 1 \le q \le \infty. \end{cases} \tag{2.19}$$

When $p = 1$ these functions can be expressed in terms of elementary functions only when q is rational, in general. Thus

$$\sin_{1,1} x = 1 - e^{-x}, \ \cos_{1,1} x = e^{-x}, \ \sin_{1,2} x = \tanh x, \ \cos_{1,2} x = (\cosh x)^{-2}. \tag{2.20}$$

Note that the area A (measured in the usual way) enclosed by the p-circle $|x|^p + |y|^p = 1$ is given by

$$A = 2p^{-1}(\Gamma(1/p))^2/\Gamma(2/p) = \pi_{p',p}. \tag{2.21}$$

To establish this, note that

$$A = 4\int\int dxdy,$$

where the integration is over all those non-negative values of x and y such that $x^p + y^p \le 1$. The change of variable $x = w^{1/p}$, $y = z^{1/p}$ shows that

$$A = 4p^{-2}\int\int w^{1/p-1}z^{1/p-1}dwdz,$$

where now the integration is taken over the set $w \geq 0, z \geq 0, w+z \leq 1$. By a result of Dirichlet (see [121], 12.5),

$$A = \frac{4(\Gamma(1/p))^2}{p^2\Gamma(2/p)} \int_0^1 \tau^{2/p-1} d\tau,$$

from which (2.21) follows.

Moreover,

$$\int_0^1 (\sin_{p,q}(\pi_{p,q}x/2))^q \, dx = p'/(p'+q) \text{ if } p,q \in (1,\infty). \tag{2.22}$$

To establish this, observe that, with the above integral denoted by I,

$$I = \frac{2}{\pi_{p,q}} \int_0^{\pi_{p,q}/2} (\sin_{p,q} y)^q \, dy,$$

so that the substitution $z = \sin_{p,q} y$ gives

$$\begin{aligned} I &= \frac{2}{\pi_{p,q}} \int_0^1 z^q (1-z^q)^{-1/p} dz = \frac{2}{q\pi_{p,q}} \int_0^1 t^{1/q}(1-t)^{-1/p} dt \\ &= \frac{2}{\pi_{p,q}} B(1/p', 1+1/q) = \frac{\Gamma(1/p'+1/q)}{\Gamma(1/q)} = \frac{p'}{q+p'}. \end{aligned}$$

Since $\left|\cos_{p,q} x\right|^p = 1 - \left|\sin_{p,q} x\right|^q$ we also have

$$\int_0^1 (\cos_{p,q}(\pi_{p,q}x/2))^p \, dx = q/(p'+q) \text{ if } p,q \in (1,\infty). \tag{2.23}$$

As shown in [92], it is interesting to compute the length $L_{p'}$ of the unit p'-circle, measured by means of the l_p metric on the plane. This is

$$L_{p'} = 4\int_0^{\pi_{p'}/2} \left(\left|x'(t)\right|^p + \left|y'(t)\right|^p\right)^{1/p} dt,$$

where $x(t) = \cos_{p'} t$ and $y(t) = \sin_{p'} t$. Routine computations plus the use of (2.6) (with p replaced by p') show that

$$L_{p'} = 4\int_0^1 (1-z^{p'})^{-1/p} dz = 2\pi_{p,p'} = \frac{4\,(\Gamma(1/p'))^2}{p'\Gamma(2/p)}.$$

In [92] it is observed that the p'-circle has an isoperimetric property, namely that among all closed curves with the same p-length, the p'-circle encloses the largest area. Since the area A enclosed by the p'-circle $|x|^{p'} + |y|^{p'} = R^{p'}$ is $\pi_{p,p'}R^2$ and the p-length of this p'-circle is $2\pi_{p,p'}R$, we have the isoperimetric inequality

$$L_{p'}^2 \geq 4\pi_{p,p'}A,$$

which reduces to the more familiar $L_2^2 \geq 4\pi A$ when $p = 2$.

As might be expected, there are connections between the generalised trigonometric functions we have been discussing and some functions from classical analysis. For example, consider the incomplete Beta function $I(\cdot; a, b)$, defined for any positive a and b by

$$I(x;a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt,\ x \in [0,1];$$

see, for example, [1, 26.5.1]. The change of variable $u = t^q$ in (2.15) shows that

$$F_{p,q}(x) = q^{-1} \int_0^{x^q} u^{-1/q'}(1-u)^{-1/p} du = q^{-1}B(1/q, 1/p')I(x^q; 1/q, 1/p'),$$

and so, by (2.13),

$$\sin_{p,q}^{-1}(x) = F_{p,q}(x) = \frac{1}{2}\pi_{p,q} I(x^q; 1/q, 1/p'),\ x \in [0,1]. \tag{2.24}$$

Moreover, since the incomplete Beta function is related to the hypergeometric function F by

$$I(x;a,b) = \frac{x^a}{aB(a,b)} F(a, 1-b; a+1; x)$$

(see [1, 6.6.2]), we have

$$\sin_{p,q}^{-1}(x) = xF(1/q, 1/p; 1+1/q; x^q),\ x \in [0,1]. \tag{2.25}$$

Since

$$I(x;a,b) = \frac{x^a(1-x)^b}{aB(a,b)} \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(a+b, n+1)} x^{n+1} \right\},\ x \in (0,1),$$

(see, for example, [1, 26.5.9]), we have

$$\sin_{p,q}^{-1}(x) = x(1-x^q)^{1/p'} \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(1+1/q, n+1)}{B(1/q+1/p', n+1)} x^{q(n+1)} \right\},\ x \in (0,1). \tag{2.26}$$

We can also use the well-known fact that

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{x^n}{n!}$$

to obtain the expansion

$$\sin_{p,q}^{-1}(x) = x\sum_{n=0}^{\infty}\frac{\Gamma(n+1/p)}{(qn+1)\Gamma(1/p)}\frac{x^{nq}}{n!},\ x \in (0,1). \tag{2.27}$$

From (2.27) it is possible to obtain a series expansion for $\sin_{p,q}(x)$ in the form $x\sum_{n=0}^{\infty} a_n x^{qn}$, but we leave this delightful task to the intrepid reader, who is urged to show that if $x \in [0, \pi_p/2)$, then

$$\sin_p x = x - \frac{1}{p(p+1)}x^{p+1} - \frac{(p^2-2p-1)}{2p^2(p+1)(2p+1)}x^{2p+1} + \ldots .$$

Finally, we consider various integrals involving the p-trigonometric functions.

Proposition 2.3. *For all* $x \in (0, \pi_p/2)$,

$$\int \cos_p x dx = \sin_p x,\ p\int \cos_p^p x dx = (p-1)x + \sin_p x \cos_p^{p-1} x,$$

$$(p-1)\int \sin_p^{p-1} x dx = -\cos_p^{p-1} x,\ \int \tan_p^p x dx = \tan_p x - x$$

and

$$\int \sin_p x dx = \frac{1}{2}\sin_p^2 x F(1/p, 2/p; 1+2/p; \sin_p^p x).$$

Proof. Apart from the last integral, these follow directly from the definitions. To obtain the final result, make the substitution $u = \sin_p x$, note that

$$\int \sin_p x dx = \int u(1-u^p)^{-1/p} du = \int u\sum_{n=0}^{\infty}\frac{\Gamma(n+1/p)}{\Gamma(1/p)}\frac{u^{pn}}{n!}du,$$

integrate, and then write the resulting series in terms of the hypergeometric function. □

For definite integrals we note the following elementary results.

Proposition 2.4. *Let* $k, l > 0$. *Then*

$$\int_0^{\pi_p/2} \sin_p^k x dx = \frac{1}{p}B\left(\frac{k+1}{p}, \frac{1}{p'}\right),\ \int_0^{\pi_p/2} \cos_p^k x dx = \frac{1}{p}B\left(\frac{1}{p}, 1+\frac{k-1}{p}\right)$$

and

$$\int_0^{\pi_p/2} \sin_p^k x \cos_p^l x dx = \frac{1}{p}B\left(\frac{k+1}{p}, 1+\frac{l-1}{p}\right).$$

These follow directly by making natural substitutions: for example, in the first integral we put $y = \sin_p x$ and then $t = y^p$. The conditions on k and l can be weakened: in the first and third equality the condition on k can be weakened to $k > -1$, while in the remaining cases the conditions $k, l > 1-p$ will do.

To illustrate the utility of Proposition 2.4 we give a result concerning the Catalan constant G, defined to be

$$G=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$

This constant plays a prominent rôle in various combinatorial identities. From the power series representation (2.27) of $\sin_p^{-1} x$ we have

$$x=\sin_p x\sum_{n=0}^{\infty}\frac{\Gamma(n+1/p)}{(np+1)\Gamma(1/p)}\frac{(\sin_p x)^{np}}{n!},\ 0<x<\frac{\pi_p}{2}.$$

Hence, with the aid of the first part of Proposition 2.4, we have

$$\int_0^{\pi_p/2}\frac{x}{\sin_p x}dx=\frac{\pi_p}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+1/p)}{n!\Gamma(1/p)}\right)^2\frac{1}{np+1}.$$

It is known that (see, for example, [63], 1.7.4)

$$\int_0^{\pi/2}\frac{x}{\sin x}dx=2G.$$

Thus the Catalan constant is expressible as

$$G=\frac{\pi}{4}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{(n!)^2 2^{2n}}\right)^2\frac{1}{2n+1}.$$

We refer to [20, 39, 89, 90] for further information and additional references concerning these functions and their applications. A fascinating account of early work on generalisations of trigonometric functions is given by Lindqvist and Peetre in [93].

2.2 Basis Properties

We have already remarked in 1.1.1 that $(\sin(n\pi\cdot))_{n\in\mathbb{N}}$ is a basis in $L_q(0,1)$ for any $q\in(1,\infty)$. It is natural to ask whether the functions $\sin_p(n\pi\cdot)$ have a similar property: the answer, given in [9], is that they do, at least if p is not too close to 1, and we now give an account of this result. For simplicity the action will take place in $L_q(0,1)$ rather than $L_q(a,b)$, and for this reason we introduce the functions $f_{n,p}$ defined by

$$f_{n,p}(t)=\sin_p(n\pi_p t)\ (n\in\mathbb{N},1<p<\infty,t\in\mathbb{R}). \tag{2.28}$$

When $p=2$ these functions are simply the usual sine functions, and we write

$$e_n(t) = f_{n,2}(t) = \sin(n\pi t). \tag{2.29}$$

Since each $f_{n,p}$ is continuous on $[0,1]$ it has a Fourier sine expansion:

$$f_{n,p}(t) = \sum_{k=1}^{\infty} \widehat{f_{n,p}}(k)\sin(k\pi t),\ \widehat{f_{n,p}}(k) = 2\int_0^1 f_{n,p}(t)\sin(k\pi t)dt. \tag{2.30}$$

From the symmetry of $f_{1,p}$ about $t = 1/2$ it follows that $\widehat{f_{1,p}}(k) = 0$ when k is even and that

$$\begin{aligned}\widehat{f_{n,p}}(k) &= 2\int_0^1 f_{1,p}(nt)\sin(k\pi t)dt = 2\sum_{m=1}^{\infty}\widehat{f_{1,p}}(m)\int_0^1 \sin(k\pi t)\sin(mn\pi t)dt \\ &= \begin{cases} \widehat{f_{1,p}}(m) & \text{if } mn = k \text{ for some odd } m, \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.31}$$

For brevity put $\tau_m(p) = \widehat{f_{1,p}}(m)$. As all the Fourier coefficients of the $f_{n,p}$ may be expressed in terms of the $\tau_m(p)$, we concentrate on the behaviour of these numbers, beginning with their decay properties as $m \to \infty$. For even m, $\tau_m(p) = 0$. If m is odd, integration by parts and the substitution $s = \cos_p(\pi_p t)$ show that

$$\begin{aligned}\tau_m(p) &= 4\int_0^{1/2} f_{1,p}(t)\sin(m\pi t)dt = \frac{4\pi_p}{m\pi}\int_0^{1/2}\cos_p(\pi_p t)\cos(m\pi t)dt \\ &= -\frac{4\pi_p}{m^2\pi^2}\int_0^{1/2}\sin(m\pi t)\frac{d}{dt}\cos_p(\pi_p t)dt \\ &= \frac{4\pi_p}{m^2\pi^2}\int_0^1 \sin\left(\frac{m\pi}{\pi_p}\cos_p^{-1} s\right)ds.\end{aligned} \tag{2.32}$$

In a similar way we have, for odd m,

$$\tau_m(p) = \frac{4}{m\pi}\int_0^1 \cos\left(\frac{m\pi}{\pi_p}\sin_p^{-1} s\right)ds. \tag{2.33}$$

From (2.32) we obtain the estimate

$$|\tau_m(p)| \le 4\pi_p/(\pi m)^2 \ (m \text{ odd}). \tag{2.34}$$

Next we consider the dependence of $\sin_p(n\pi_p t)$ on p.

Proposition 2.5. *Suppose that* $1 < p < q < \infty$. *Then the function* f *defined by*

$$f(t) = \frac{\sin_q^{-1}(t)}{\sin_p^{-1}(t)}$$

is strictly decreasing on $(0,1)$.

Proof. Let

$$g(t) = \frac{(1-t^q)^{1/q}}{(1-t^p)^{1/p}} \ (0 < t < 1).$$

For all $t \in (0,1)$,

$$g'(t) = g(t)\left\{\frac{-t^{q-1}}{1-t^q} + \frac{t^{p-1}}{1-t^p}\right\} = \frac{(t^p - t^q)g(t)}{t(1-t^q)(1-t^p)} > 0.$$

Put

$$G(t) = \sin_p^{-1}(t) - g(t)\sin_q^{-1}(t)$$

and observe that

$$G'(t) = -(\sin_q^{-1} t)g'(t) < 0 \text{ in } (0,1).$$

Hence $G(t) < 0$ in $(0,1)$, so that

$$f'(t) = \frac{G(t)}{(\sin_q^{-1} t)^2(1-t^q)^{1/q}} < 0 \text{ in } (0,1).$$ □

From this we immediately have

Corollary 2.1. *(i) If* $1 < p < q < \infty$, *then*

$$1 > \frac{\sin_q^{-1}(t)}{\sin_p^{-1}(t)} \geq \frac{\pi_q}{\pi_p} \textit{ in } (0,1].$$

(ii) If $1 < p \leq q < \infty$, *then*

$$\sin_p^{-1}(t) \geq \sin_q^{-1}(t) \textit{ and } \frac{1}{\pi_q}\sin_q^{-1}(t) \geq \frac{1}{\pi_p}\sin_p^{-1}(t) \textit{ in } [0,1].$$

(iii) If $1 < p \leq q < \infty$, *then*

$$\sin_p(\pi_p t) \geq \sin_q(\pi_q t) \textit{ in } \ [0,1/2].$$

The following analogue of the classical Jordan inequality will also be useful.

Proposition 2.6. *Let* $1 < p < \infty$. *For all* $\theta \in (0, \pi_p/2]$,

$$\frac{2}{\pi_p} \leq \frac{\sin_p \theta}{\theta} < 1.$$

Proof. Change of variable shows that

$$\sin_p^{-1} x = x\int_0^1 (1 - x^p s^p)^{-1/p} ds,$$

and so

$$\theta = (\sin_p \theta) \int_0^1 (1-(\sin_p \theta)^p s^p)^{-1/p} ds.$$

Since

$$1 \leq \int_0^1 (1-(\sin_p \theta)^p s^p)^{-1/p} ds \leq \frac{\pi_p}{2}$$

for all $\theta \in (0, \pi_p/2]$, the result follows. □

Corollary 2.2. *For all $p \in (1,\infty)$ and all $t \in (0,1/2)$,*

$$\sin_p(\pi_p t) > 2t.$$

Proof. By Proposition 2.6, $\sin_p \theta > 2\theta/\pi_p$ if $0 < \theta < \pi_p/2$. Now put $\theta = \pi_p t$. □

Given any function f on $[0,1]$, we extend it to a function $\widetilde{f}$ on $\mathbb{R}_+ := [0,\infty)$ by setting

$$\widetilde{f}(t) = -\widetilde{f}(2k-t) \text{ for } t \in [k,k+1], k \in \mathbb{N}. \tag{2.35}$$

With this understanding, we define maps $M_m : L_q(0,1) \to L_q(0,1)$ $(1 < q < \infty)$ by

$$M_m g(t) = \widetilde{g}(mt),\ m \in \mathbb{N},\ t \in (0,1). \tag{2.36}$$

Note that $M_m e_n = e_{mn}$.

Lemma 2.1. *For all $m \in \mathbb{N}$ and all $q \in (1,\infty)$ the map $M_m : L_q(0,1) \to L_q(0,1)$ is isometric and linear.*

Proof. Let $g \in L_q(0,1)$. Then

$$\begin{aligned}\int_0^1 |M_m g(t)|^q \, dt &= m^{-1} \int_0^m |\widetilde{g}(s)|^q \, ds = m^{-1} \sum_{k=1}^m \int_{k-1}^k |\widetilde{g}(s)|^q \, ds \\ &= m^{-1} \sum_{k=1}^m \int_{k-1}^k |g(s)|^q \, ds = \int_0^1 |g(s)|^q \, ds.\end{aligned}$$

□

The maps M_m are introduced because they help to construct a linear homeomorphism T of $L_q(0,1)$ onto itself that maps each e_n to $f_{n,p}$: once this is done it will follow from general considerations that the $f_{n,p}$ form a basis of $L_q(0,1)$. The map T is defined by

$$Tg(t) = \sum_{m=1}^{\infty} \tau_m M_m g(t). \tag{2.37}$$

Lemma 2.2. *Let $p,q \in (1,\infty)$. The map T is a bounded linear map of $L_q(0,1)$ to itself with $\|T\| \leq \pi_p/2$. For all $n \in \mathbb{N}$, $Te_n = f_{n,p}$.*

Proof. From (2.31), (2.34) and Lemma 2.1 we see that

$$\|T\| \le \sum_{m=1}^{\infty} \frac{4\pi_p}{(2m-1)^2\pi^2} = \pi_p/2.$$

A second application of (2.31) shows that

$$Te_n = \sum_{m=1}^{\infty} \tau_m e_{mn} = \sum_{m=1}^{\infty} \widehat{f_{1,p}}(m) e_{mn} = \sum_{k=1}^{\infty} \widehat{f_{n,p}}(k) e_k = f_{n,p}. \qquad \square$$

Lemma 2.3. *There exists* $p_0 \in (1,2)$ *such that if* $p > p_0$, *then for all* $q \in (1,\infty)$, $T : L_q(0,1) \to L_q(0,1)$ *has a bounded inverse.*

Proof. Since M_1 is the identity map id, we have from (2.31) and Lemma 2.1 that

$$\|T - \tau_1 id\| \le \sum_{j=1}^{\infty} \left|\tau_{2j+1}(p)\right|,$$

and so the invertibility of T will follow from Theorem II.1.2 of [123] if we can show that

$$\sum_{j=1}^{\infty} \left|\tau_{2j+1}(p)\right| < \left|\tau_1(p)\right|. \tag{2.38}$$

From (2.34) we have, for all $p \in (1,\infty)$,

$$\sum_{j=1}^{\infty} \left|\tau_{2j+1}(p)\right| \le \frac{4\pi_p}{\pi^2}\left(\frac{\pi^2}{8} - 1\right). \tag{2.39}$$

To estimate $|\tau_1(p)|$, note that by Corollary 2.2,

$$\tau_1(p) = 4\int_0^{1/2} \sin_p(\pi_p t)\sin(\pi t)\,dt > 4\int_0^{1/2} 2t\sin(\pi t)\,dt = 8/\pi^2,$$

from which (2.38) follows if $2 \le p < \infty$ since $\pi_p \le \pi$.

If $1 < p < 2$, then the monotonic dependence of $\sin_p(\pi_p t)$ on p given by Corollary 2.1 (iii) shows that

$$\tau_1(p) > 4\int_0^{1/2} \sin^2(\pi t)\,dt = 1.$$

Now define p_0 by

$$\pi_{p_0} = \frac{\pi^2}{4} \Big/ \left(\frac{\pi^2}{8} - 1\right).$$

Then if $p > p_0$,

$$\frac{4\pi_p}{\pi^2}\left(\frac{\pi^2}{8}-1\right)<1,$$

and again we have (2.38).

We summarise these results in the following theorem.

Theorem 2.1. *The map T is a homeomorphism of $L_q(0,1)$ onto itself for every $q\in(1,\infty)$ if $p_0<p<\infty$, where p_0 is defined by the equation*

$$\pi_{p_0}=\frac{2\pi^2}{\pi^2-8}. \tag{2.40}$$

Remark 2.1. Numerical solution of (2.40) shows that p_0 is approximately equal to 1.05.

Theorem 2.2. *Let $p\in(p_0,\infty)$ and $q\in(1,\infty)$. Then the family $(f_{n,p})_{n\in\mathbb{N}}$ forms a Schauder basis of $L_q(0,1)$ and a Riesz basis of $L_2(0,1)$.*

Proof. Since the e_n form a basis of $L_q(0,1)$ and T is a linear homeomorphism of $L_q(0,1)$ onto itself with $Te_n=f_{p,n}$ $(n\in\mathbb{N})$, it follows from [73], p. 75 or [114], Theorem 3.1, p. 20 that the $f_{n,p}$ form a Schauder basis of $L_q(0,1)$. When $q=2$ the argument is similar and follows [67], Sect. VI.2. □

The condition $p>p_0>1$ in this theorem arises from the techniques used in the proof: a discussion of this is given in [20]. Whether the result remains true for all $p>1$ appears to be unknown at the moment.

Notes

Note 2.1. As the literature contains various different definitions of the $\sin_p$ and $\cos_p$ functions, confusion about the nature of such functions is possible. Our choice was largely motivated by the wish to have available the identity $\left|\sin_p x\right|^p+\left|\cos_p x\right|^p=1$, while other authors attached greater importance to different properties. Power series expansions for his versions of $\sin_p$, $\cos_p$ and $\tan_p$ are given by Linqvist [90]; see also the detailed work in this direction on related functions by Peetre [104]. No sensible addition formulae (e.g. for $\sin_p(x+y)$) seem to be known. Further details of properties of p-trigonometric functions are given in [20].

Note 2.2. The only work on the basis properties of the $\sin_p$ functions of which we are aware is that of [9]. Our treatment gives the modification of their proof presented in [20], which in particular seals a gap in the proof of Corollary 2.1(iii) given in [9].

Completeness properties of certain function sequences of the form $\{f(nx)\}_{n\in\mathbb{N}}$ have been investigated by Bourgin ([16]; see also [17]) in an L_2 setting and by Szász [119] in the context of L_r. However, these papers require properties, such as orthogonality or specified behaviour of the Fourier coefficients of f, that are not available when $f=\sin_p$.

Chapter 3
The Laplacian and Some Natural Variants

Our focus in this chapter is largely on the p-Laplacian. The theory of Chap. 1 concerning the representation of compact linear maps is used to establish the existence of a countable family of certain types of weak solutions of the Dirichlet eigenvalue problem for the p-Laplacian, with associated eigenvalues. When the underlying space domain is a bounded interval in the real line more direct methods are available: we give an account of the work of [39] which leads to the representation in terms of p-trigonometric functions of the eigenfunctions of the one-dimensional p-Laplacian under a variety of initial or boundary conditions.

3.1 The Laplacian

The trigonometric functions have natural connections with the Laplace operator. To illustrate this, let Q be the open cube $(-1,1)^n$ in $\mathbb{R}^n$, let V be the set of all vertices of Q, put $\partial Q' = \partial Q \backslash V$ and let $\partial/\partial\nu$ denote differentiation along the normal outwards from Q at points of $\partial Q'$. Put

$$\Delta = \sum_{j=1}^{n} D_j^2 \ (D_j = \partial/\partial x_j) \text{ and } D(A) = \left\{ f \in C^\infty(\overline{Q}) : f = 0 \text{ on } \partial Q \right\},$$

and let $A : D(A) \ (\subset L_2(Q)) \to L_2(Q)$ be defined by $Af = -\Delta f \ (f \in D(A))$; A is a symmetric operator acting in $L_2(Q)$. Denote by $-\Delta_D$ the closure of A with respect to the graph norm; this is a self-adjoint map, the Friedrichs extension of A, and is called the Dirichlet Laplacian on Q. Similarly, let

$$D(B) = \left\{ f \in C^\infty(\overline{Q}) : \partial f/\partial\nu = 0 \text{ on } \partial Q' \right\},$$

define $B : D(B) \to L_2(Q)$ by $Bf = -\Delta f \ (f \in D(B))$ and let $-\Delta_N$ be the closure of B. The self-adjoint map $-\Delta_N$ is called the Neumann Laplacian on Q.

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The eigenvalues and eigenvectors of $-\Delta_D$ and $-\Delta_N$ may be computed explicitly. In fact, routine separation of variables arguments show that the (normalised) eigenvectors of $-\Delta_D$ are the functions Φ_α $(\alpha \in \mathbb{N}^n)$, where

$$\Phi_\alpha(x) = \prod_{j=1}^{n} \phi_{\alpha_j}(x_j)$$

and

$$\phi_l(y_k) = \begin{cases} \cos(l\pi y_k/2), & l \text{ odd}, \\ \sin(l\pi y_k/2), & l \text{ even}. \end{cases}$$

These functions form an orthonormal basis of $L_2(Q)$. The corresponding eigenvalues are

$$\lambda_\alpha = (\pi/2)^2 \sum_{j=1}^{n} \alpha_j^2.$$

For $-\Delta_N$ the (normalised) eigenvectors are the functions Ψ_α $(\alpha \in \mathbb{N}_0^n)$, where

$$\Psi_\alpha(x) = \prod_{j=1}^{n} \psi_{\alpha_j}(x_j)$$

and

$$\psi_l(y_k) = \begin{cases} \sin(l\pi y_k/2), & l \text{ odd}, \\ \cos(l\pi y_k/2), & l \text{ even. } l \neq 0, \\ 1/\sqrt{2}, & l = 0. \end{cases}$$

The related eigenvalues are λ_α $(\alpha \in \mathbb{N}_0^n)$. For further details of this and allied material we refer to [41], Chap. XI.

3.2 The *p*-Laplacian

We begin with the one-dimensional situation. Let $a, b \in \mathbb{R}$ with $a < b$, put $I = (a, b)$ and let $1 \leq p < \infty$. As we have already seen in 1.2, the Sobolev space $W_p^1(I)$ is the Banach space of all those (equivalence classes of) functions u in $L_p(I)$ that have first-order distributional derivatives u' belonging to $L_p(I)$, endowed with the norm

$$\left\| u \mid W_p^1(I) \right\| := \left(\left\| u \mid L_p(I) \right\|^p + \left\| u' \mid L_p(I) \right\|^p \right)^{1/p}. \tag{3.1}$$

By $\overset{0}{W}{}_p^1(I)$ we shall mean the completion of $C_0^\infty(I)$, the space of all infinitely differentiable functions with compact support in I, with respect to the norm

$$\left\| u \mid \overset{0}{W}{}_p^1(I) \right\| := \left\| u' \mid L_p(I) \right\|. \tag{3.2}$$

Evidently $\overset{0}{W}{}_p^1(I)$ is a closed subspace of $W_p^1(I)$ and the norm defined by (3.2) is equivalent to $\left\|\cdot \mid W_p^1(I)\right\|$ on this subspace, in view of the Friedrichs inequality

$$\left\|u \mid L_p(I)\right\| \leq (b-a)\left\|u' \mid L_p(I)\right\|, \; u \in \overset{0}{W}{}_p^1(I). \tag{3.3}$$

Each element of $\overset{0}{W}{}_p^1(I)$ has a representative that is absolutely continuous on I and vanishes at the endpoints a and b. For all these facts we refer to [41], Chap. V.

Now put $X = \overset{0}{W}{}_p^1(I)$ and consider the problem of minimising the Rayleigh quotient

$$R(u) := \frac{\left\|u' \mid L_p(I)\right\|}{\left\|u \mid L_p(I)\right\|} \tag{3.4}$$

over $X \backslash \{0\}$. A standard way of dealing with this is as follows. The question amounts to minimising

$$E(v) := \int_a^b \left|v'(x)\right|^p dx \tag{3.5}$$

over X subject to the constraint

$$F(v) := \int_a^b |v(x)|^p \, dx - 1 = 0. \tag{3.6}$$

To see that this minimum really exists, put

$$\lambda = \inf E(v),$$

where the inf is taken over all $v \in X$ with $F(v) = 1$, and let (v_m) be a minimising sequence:

$$\left\|v_m \mid L_p(I)\right\| = 1, \int_a^b \left|v_m'(x)\right|^p dx = \lambda + \varepsilon_m, \varepsilon_m \to 0 \text{ as } m \to \infty.$$

Then (v_m) is bounded in X and so there is a subsequence, again denoted by (v_m) for convenience, that converges weakly in X, to u, say. Since X is compactly embedded in $L_p(I)$, $v_m \to u$ in $L_p(I)$: $\left\|u \mid L_p(I)\right\| = 1$ and so $u \neq 0$. Moreover, since $v_m \rightharpoonup u$ in X,

$$\|u \mid X\|^p \leq \liminf_{m \to \infty} \|v_m \mid X\|^p = \lambda.$$

Hence $\int_a^b |u'(x)|^p \, dx = \lambda$; and as $u \in X \backslash \{0\}$ it cannot be constant. Thus $\lambda > 0$ and so the infimum is attained at u and is positive. The problem of minimising $R(v)$, or equivalently that of minimising $E(v)$ subject to the constraint $F(v) = 0$, can be analysed by use of the infinite-dimensional version of the Lagrange multiplier theorem (see [34], Theorem 26.1, p. 333). This enables us to conclude that there is a real number μ such that for all $h \in X$,

$$0 = \langle h, \operatorname{grad} E(u) - \mu \operatorname{grad} F(u) \rangle_X = p \int_a^b \left\{ |u'|^{p-2} u'h' - \mu |u|^{p-2} uh \right\} dx. \quad (3.7)$$

This means that u is a weak solution (an eigenfunction) of the nonlinear eigenvalue problem

$$-\left(|u'|^{p-2} u' \right)' = \mu |u|^{p-2} u, u(a) = u(b) = 0, \quad (3.8)$$

in which μ is the corresponding eigenvalue. The choice $h = u$ in (3.7) shows that

$$\mu = \mu \int_a^b |u|^p \, dx = \int_a^b |u'|^p \, dx = \lambda.$$

Hence the infimum λ is an eigenvalue of (3.8). It is the least such eigenvalue: for if $v \in X$ is an eigenfunction, with corresponding eigenvalue λ_1, then we may put $h = v$ in (3.7) and obtain

$$\lambda_1 \int_a^b |v|^p \, dx = \int_a^b |v'|^p \, dx,$$

so that if $w := v / \| v \mid L_p(I) \|$, then $\| w \mid L_p(I) \| = 1$ and $\lambda_1 = \int_a^b |w'|^p \, dx$. Thus $\lambda_1 \geq \lambda$.

In a corresponding way, given any $p, q \in (1, \infty)$, the problem of minimising

$$\| u' \mid L_p(I) \| / \| u \mid L_q(I) \|$$

over $X \backslash \{0\}$ gives rise to the eigenvalue problem

$$-\left(|u'|^{p-2} u' \right)' = \lambda |u|^{q-2} u, \; u(a) = u(b) = 0. \quad (3.9)$$

However, as we now show, these results can be obtained more quickly by using the theory developed in Sect. 1.3. We take $X = \overset{0}{W}{}^1_p(I)$ as above and $Y = L_q(I)$, where $p, q \in (1, \infty)$. The natural embedding id: $X \to Y$ is compact, for by [82], Remarks 5.8.4 (i), X is compactly embedded in $C(\overline{\Omega})$ and hence in Y. Evidently both X and Y are reflexive and strictly convex; clearly Y^* is also strictly convex. Since $\| \cdot \mid X \|$ is Gâteaux-differentiable on $X \backslash \{0\}$ it follows from Proposition 1.8 that X^* is strictly convex. With the maps $\widetilde{J}_X$ and $\widetilde{J}_Y$ defined as in Sect. 1.3, direct verification shows that

$$\widetilde{J}_Y u = \|u\|_q^{-(q-1)} |u|^{q-2} u, \; u \in Y \backslash \{0\}.$$

Moreover, in the sense of distributions, we have for all $u \in X$,

$$\widetilde{J}_X u = - \| u \mid X \|^{-(p-1)} \Delta_p u, \text{ where } \Delta_p u = \left(|u'|^{p-2} u' \right)'.$$

This follows since for all $u \in X$,

$$\left\langle u, -\|u \mid X\|^{-(p-1)} \Delta_p u \right\rangle_X = \|u \mid X\|^{-(p-1)} \int_I u' \cdot \left|u'\right|^{p-2} u' dx = \|u \mid X\|.$$

Proposition 1.11 now gives the existence of a function $u_1 \in X$, with $\|u_1 \mid Y\| = \|\mathrm{id}\| = \lambda_1$, such that for all $v \in X$,

$$\left\langle v, \widetilde{J}_Y u_1 \right\rangle_Y = \lambda_1 \left\langle v, \widetilde{J}_X u_1 \right\rangle_X,$$

which amounts to

$$\int_I v \left|u_1\right|^{q-2} u_1 dx = \lambda_1^q \int_I v' \cdot \left|u'\right|^{p-2} u' dx,$$

so that u_1 is a weak solution of the Dirichlet eigenvalue problem (3.9). In fact, since id is not of finite rank, Theorem 1.7 ensures that for each $k \in \mathbb{N}$, there are an 'eigenvector' u_k and a corresponding 'eigenvalue' λ_k^{-p}, with $\lambda_k^{-p} \to \infty$ as $k \to \infty$, that satisfy

$$-\Delta_p u_k = \lambda_k^{-p} \left|u_k\right|^{p-2} u_k, \; u_k = 0 \text{ on } \partial I, \tag{3.10}$$

in the sense that for all $v \in X_k$,

$$\int_I v \left|u_k\right|^{q-2} u_k dx = \lambda_k^q \int_I v' \cdot \left|u_k'\right|^{p-2} u_k' dx. \tag{3.11}$$

We shall refer to u_k as a k-weak solution of (3.10): note that when $k = 1$, all functions in $X_1 = X$ are allowed as test functions, while for general $k > 1$ the test functions have to be taken from $X_k \subset X$.

We also observe that problem (3.8) studied earlier in this section can be handled by the methods of 1.3 on taking $X = L_p(a,b)$ and T to be the Hardy operator given by $(Tf)(x) = \int_a^x f(t)dt$. We shall study such operators in detail in the next Chapter.

The techniques just used, arising from Sect. 1.3, work equally well for the higher-dimensional p-Laplacian. Thus let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$ and take X to be $\overset{0}{W}{}_p^1(\Omega)$, the closure in $W_p^1(\Omega)$ of the set $C_0^\infty(\Omega)$ of all infinitely differentiable functions with compact support in Ω; define the norm on X by

$$\|u\|_X = \left(\int_\Omega \sum_{j=1}^n \left|D_j u\right|^p dx \right)^{1/p}.$$

Because of the Friedrichs inequality (see [41], Theorem V.3.22), this norm is equivalent to the norm on X inherited from $W_p^1(\Omega)$. Let $Y = L_p(\Omega)$, $T = \mathrm{id}\colon X \to Y$; id is compact. As in the one-dimensional case it follows that both X and Y are reflexive and strictly convex, with strictly convex duals, and that

$$\widetilde{J}_Y u = \|u\|_p^{-(p-1)} |u|^{p-2} u, \widetilde{J}_X u = -\|u\|_X^{-(p-1)} \Delta_p u \text{ (in the distributional sense)},$$

where $\|\cdot\|_p$ is the usual norm on $L_p(\Omega)$ and now

$$\Delta_p u = \sum_{j=1}^{n} D_j \left(\left| D_j u \right|^{p-2} D_j u \right).$$

Proposition 1.11 now shows that there exists $u_1 \in X$, with $\|u_1\|_Y = \|\text{id}\| = \lambda_1$, such that

$$\left\langle v, \widetilde{J}_Y u_1 \right\rangle_Y = \lambda_1 \left\langle v, \widetilde{J}_X u_1 \right\rangle_X \text{ for all } v \in X,$$

so that

$$\int_\Omega v \left| u_1 \right|^{p-2} u_1 dx = \lambda_1^p \int_\Omega \sum_{j=1}^{n} (D_j v) \left| D_j u_1 \right|^{p-2} D_j u_1 dx.$$

Hence u_1 is a weak solution of the Dirichlet eigenvalue problem

$$-\Delta_p u_1 = \lambda_1^{-p} \left| u_1 \right|^{p-2} u_1,\ u_1 = 0 \text{ on } \partial\Omega.$$

Again Theorem 1.7, ensures the existence of a sequence (u_k), where each u_k is a k-weak solution of this problem, using the same terminology as for the one-dimensional case above, and a corresponding sequence (λ_k^{-p}) of eigenvalues with $\lambda_k^{-p} \to \infty$ as $k \to \infty$. Results of this kind are known for the eigenvalues obtained by the Lyusternik–Schnirelmann procedure (see, for example, [26] and [65], where a slightly different form of the p-Laplacian is considered and where for each k the eigenvector corresponding to the kth eigenvalue is a classical weak solution, not merely a k-weak solution; see the Notes at the end of this Chapter), but the simplicity of the present approach has its attractions. Moreover, information about the growth of the λ_k^{-p} can be obtained with little additional effort. In fact, if we denote by $c_k(S)$ and $x_k(S)$ the kth Gelfand and Weyl numbers respectively of a map S (see Definition 5.4 below and also [41], pp. 72 and 77), and let $I : W_p^1(\Omega) \to L_p(\Omega)$ be the natural embedding, then from the definitions it follows that

$$\lambda_k \geq c_k(id) \geq c_k(I) \geq x_k(I).$$

From [80], Theorem 3.c.5 and Remark 3.c.7 (1), we see that $x_k(I) \geq ck^{-1/n}$, where c is a positive constant independent of k. Hence $\lambda_k \geq ck^{-1/n}$, and so the eigenvalues λ_k^{-p} of the Dirichlet problem are $0(k^{p/n})$.

If instead we suppose that $1 < p < n$, $q \in \left(1, \frac{np}{n-p}\right)$ and take $X = \overset{0}{W}{}_p^1(\Omega)$, $Y = L_q(\Omega)$, $T = \text{id}\colon X \to Y$, with Ω as before, then as id is compact, the same procedure establishes the existence of k-weak solutions v_k and corresponding eigenvalues μ_k $(k \in \mathbb{N})$ of the p,q-Laplacian Dirichlet eigenvalue problem

$$-\Delta_p v = \mu \left| v \right|^{q-2} v,\ v = 0 \text{ on } \partial\Omega.$$

3.3 Eigenfunctions of the p-Laplacian, $n = 1$

Here we present the arguments of [39] for dealing with eigenfunctions and eigenvalues of the nonlinear differential equation in (3.9) under a variety of initial or boundary conditions For simplicity we shall write

$$(s)_{(r)} = |s|^{r-2} s \; (s \in \mathbb{R}\backslash\{0\}), \; (0)_{(r)} = 0, \tag{3.12}$$

whenever $r \in (1, \infty)$. With this notation, (3.9) may be written in the form

$$-\left((u')_{(p)}\right)' = \lambda (u)_{(q)}. \tag{3.13}$$

Note that

$$v = (u)_{(p)} \text{ is equivalent to } u = (v)_{(p')}. \tag{3.14}$$

To begin with, we shall consider the initial-value problem

$$\left.\begin{aligned} \left((u')_{(p)}\right)' + \lambda (u)_{(q)} &= 0, \\ u(t_0) = a, \; u'(t_0) \; &= b, \end{aligned}\right\} \tag{3.15}$$

where a, b and t_0 are arbitrary real numbers and $p, q \in (1, \infty)$.

Theorem 3.1. *Given any $\lambda \geq 0$, the problem (3.15) has a unique solution $u \in C^1(\mathbb{R})$ for which $(u')_{(p)} \in C^1(\mathbb{R})$.*

Proof. First suppose that $\lambda = 0$. Then (3.15) immediately gives

$$(u')_{(p)} = (b)_{(p)},$$

so that $u'(t) = b$ and the unique solution of the problem is

$$u(t) = b(t - t_0) + a, \; t \in \mathbb{R}.$$

Now let $\lambda > 0$. We may assume without loss of generality that $t_0 = 0$. The problem (3.15) is equivalent to

$$(u'(t))_{(p)} = -\lambda \int_0^t (u(s))_{(q)} ds + (b)_{(p)} := F(t, u), \; u(0) = a, \tag{3.16}$$

and thus also to

$$u'(t) = (F(t, u))_{(p')}, \; u(0) = a. \tag{3.17}$$

We first prove the existence of a C^1 solution u of (3.17), with $(u')_{(p)} \in C^1$, in an interval $I_\delta = [0, \delta]$ for some small $\delta > 0$. Define a map $T{:}C(I_\delta) \to C(I_\delta)$ by

$$(Tv)(t) = \int_0^t (F(s, v))(p') ds + a := u(t), \tag{3.18}$$

for $v \in C(I_\delta)$ and $t \in I_\delta$. Then $u \in C^1(I_\delta)$ and

$$u'(t) = (F(t,v))(p'),\ u(0) = a,\ u'(0) = b. \tag{3.19}$$

For the moment, denote the norm on $C(I_\delta)$ by $\|\cdot\|$ and put $w(t) = a + tb$. Suppose that $\|v - w\| \le A$. Then

$$\|u - w\| \le \delta\,|b| + \delta\left(\lambda\delta\,(A + |a| + \delta\,|b|)_{(q)} + \left|(b)_{(p)}\right|\right)_{(p')} \le A$$

for small enough $\delta > 0$. Henceforth we suppose that such a choice of δ has been made. Denote by B the closed ball in $C(I_\delta)$ with centre w and radius A. Then it follows that $T(B) \subset B$. A routine argument shows that T is continuous on B; and given any bounded subset K of B, a standard use of the Arzelà–Ascoli theorem shows that $T(K)$ is a relatively compact subset of $C(I_\delta)$, so that T is compact. Hence by Schauder's fixed-point theorem, T has a fixed point, say u, in $B : Tu = u$. This implies that

$$u(t) = \int_0^t (F(s,u))_{(p')} ds + a,\ t \in I_\delta,$$

and so

$$u'(t) = (F(t,u))_{(p')} \text{ in } I_\delta,$$

that is

$$(u'(t))_{(p)} = F(t,u) = -\lambda \int_0^t (u(s))_{(q)} ds + (b)_{(p)},$$

which shows that

$$\left((u'(t))_{(p)}\right)' = -\lambda\,(u(t))_{(q)},\ t \in I_\delta,$$

and

$$u(0) = a,\ u'(0) = b.$$

The existence of a solution of (3.17) in any sufficiently small interval containing 0 follows by similar means.

Next, we prove that this local solution u is unique, and proceed by enumeration of cases, dealing only with $t > 0$. Note that

$$(u)_{(q)}u' = |u|^{q-2}\,uu' = \frac{1}{q}\frac{d}{dt}\left(|u|^q\right)$$

and that

$$u'\left((u')_{(p)}\right)' = \frac{1}{p'}\frac{d}{dt}\left(|u'|^p\right).$$

Multiplication of the differential equation in (3.15) by u' thus gives

$$\frac{d}{dt}\left(\frac{|u'|^p}{p'} + \lambda\frac{|u|^q}{q}\right) = 0,$$

so that

$$\frac{|u'(t)|^p}{p'} + \lambda \frac{|u(t)|^q}{q} = \frac{|b|^p}{p'} + \lambda \frac{|a|^q}{q}. \tag{3.20}$$

This holds for all t in the domain of definition of u.

(i) Suppose that $a = b = 0$. Then by (3.20), u is identically zero.

(ii) Assume that $a = 0$, $b \neq 0$ and suppose that u_1, u_2 are two local solutions corresponding to these values. Then

$$\begin{aligned}(u_1'(t))_{(p)} - (u_2'(t))_{(p)} &= \lambda \int_0^t \left\{(u_2(\tau))_{(q)} - (u_1(\tau))_{(q)}\right\} d\tau \\ &= \lambda \int_0^t \tau^{q-1} \left\{ \left(\frac{u_2(\tau)}{\tau}\right)_{(q)} - \left(\frac{u_1(\tau)}{\tau}\right)_{(q)} \right\} d\tau.\end{aligned} \tag{3.21}$$

Since $u_1(\tau)/\tau \to b$ and $u_2(\tau)/\tau \to b \neq 0$ as $\tau \to 0$, it follows that for small enough $t > 0$, both $u_1'(t)$ and $u_2'(t)$ lie in a small interval in which $(.)_{p'}$ and $(.)_q$ are C^1. We thus see from the mean-value theorem that there exists $K > 0$ such that for all small enough $t > 0$,

$$\left|u_1'(t) - u_2'(t)\right| \leq K\lambda \int_0^t \tau^{q-1} \left| \frac{u_1(\tau)}{\tau} - \frac{u_2(\tau)}{\tau} \right| d\tau.$$

Put $w = u_1 - u_2$; then for some small $\varepsilon > 0$,

$$\sup_{t \in (0,\varepsilon)} \left|w'(t)\right| \leq \frac{K\lambda \varepsilon^{q-1}}{q-1} \sup_{t \in (0,\varepsilon)} |w(t)|. \tag{3.22}$$

Since $w(t) = \int_0^t w'(\tau) d\tau$ and $w(0) = 0$ we have

$$\sup_{t \in (0,\varepsilon)} |w(t)| \leq \varepsilon \sup_{t \in (0,\varepsilon)} \left|w'(t)\right|.$$

Together with (3.22) this gives

$$\left(1 - \frac{K\lambda \varepsilon^q}{q-1}\right) \sup_{t \in (0,\varepsilon)} \left|w'(t)\right| \leq 0,$$

which is impossible, for small enough $\varepsilon > 0$, unless $u_1 = u_2$.

(iii) Suppose that $a \neq 0$ and $b = 0$. Put $v = (u')_{(p)}$, so that

$$u' = (v)_{(p')}. \tag{3.23}$$

From the differential equation satisfied by u we see that $v' = -\lambda (u)_{(q)}$, which gives

$$u = -(v')_{(q')} \lambda^{-q'+1}. \tag{3.24}$$

Equations (3.23) and (3.24) mean that v satisfies

$$\left((v')_{(q')}\right)' + \lambda^{q'-1}(v)_{(p')} = 0,$$

together with the conditions $v(0) = 0$, $v'(0) = -\lambda(a)_{(q)} \neq 0$. Hence case (iii) is reduced to (ii).

(iv) Assume that $a \neq 0$ and $b \neq 0$. As in (iii) we reduce the differential equation to the system

$$u' = (v)_{(p')},\ v' = -\lambda(u)_{(q)}. \tag{3.25}$$

In this case $u(0) \neq 0$ and $v(0) \neq 0$, so that for small enough $t > 0$, the right-hand sides of (3.25) lie in an interval in which $(.)_{(p')}$ and $(.)_{(q)}$ are of class C^1 and therefore Lipschitz. Classical uniqueness theorems (see, for example, [11], Theorem 6.3.1) now show that the solution is locally unique.

Local uniqueness is thus established in all possible cases. The boundedness of u and u' that follows from (3.20) means that the above existence arguments may be repeated arbitrarily often, with t steps of constant length, so that we also have global existence and uniqueness. □

Our next aim is to show that the solution of (3.15) can be expressed in terms of the $\sin_{p,q}$ functions introduced earlier. To do this we first study the initial-value problem

$$\left.\begin{aligned} \left((u')_{(p)}\right)' + \lambda(u)_{(q)} &= 0, \\ u(0) = 0,\ u'(0) &= \alpha, \end{aligned}\right\} \tag{3.26}$$

where $\alpha > 0$. Suppose that u is a solution of this problem and that the first positive zero of u' is at t_α. Then on $(0, t_\alpha)$ we have $u(t) > 0$ and $u'(t) > 0$, so that by (3.20),

$$\frac{u'(t)^p}{p'} + \lambda\frac{u(t)^q}{q} = \frac{\lambda R^q}{q} = \frac{\alpha^p}{p'}, \tag{3.27}$$

where $R = u(t_\alpha)$. From this we obtain

$$\begin{aligned} t &= \left(\frac{q}{\lambda p'}\right)^{1/p} \int_0^t \frac{u'(s)}{(R^q - u(s)^q)^{1/p}} ds = \left(\frac{q}{\lambda p'}\right)^{1/p} R^{1-q/p} \int_0^{u(t)/R} \frac{ds}{(1-s^q)^{1/p}} \\ &= \left(\frac{q}{\lambda p'}\right)^{1/p} R^{1-q/p}\ \sin_{p,q}^{-1}(u(t)/R). \end{aligned} \tag{3.28}$$

It follows that

$$u(t) = R\sin_{p,q}\left\{\left(\frac{\lambda p'}{q}\right)^{1/p} R^{(q-p)/p} t\right\}. \tag{3.29}$$

From (3.27) we have

$$R = \left(\frac{q}{\lambda p'}\right)^{1/q} \alpha^{p/q},$$

which gives

$$u(t) = \frac{\alpha}{A_{p,q}(\alpha,\lambda)} \sin_{p,q}\left\{A_{p,q}(\alpha,\lambda)t\right\}, \tag{3.30}$$

where

$$A_{p,q}(\alpha,\lambda) = \left(\frac{\lambda p'}{q}\right)^{1/q} \alpha^{(q-p)/q}. \tag{3.31}$$

We have thus found the unique solution u of the initial-value problem (3.26) in the interval $(0,t_\alpha)$. However, in view of the periodicity properties of $\sin_{p,q}$ this function u is actually the unique global solution of the problem, when $\alpha > 0$; if $\alpha < 0$ we merely have to replace α by $|\alpha|$ in $A_{p,q}$. These results are summarised in

Theorem 3.2. *Given any $\alpha \in \mathbb{R}$, the unique global solution of (3.26) is*

$$u(t) = \frac{\alpha}{A_{p,q}(|\alpha|,\lambda)} \sin_{p,q}\left\{A_{p,q}(|\alpha|,\lambda)t\right\},\ t \in \mathbb{R}. \tag{3.32}$$

From this it is easy to give the solution of the initial-value problem (3.15). Since the differential equation involved is autonomous, it is clear that the solution will be of the form

$$u(t) = \frac{\alpha}{A_{p,q}(|\alpha|,\lambda)} \sin_{p,q}\left\{A_{p,q}(|\alpha|,\lambda)(t-t_0)+\delta\right\},\ t \in \mathbb{R}, \tag{3.33}$$

where α and δ are to be determined in terms of a and b. To do this, notice that

$$u(t_0) = a = \frac{\alpha}{A_{p,q}(|\alpha|,\lambda)} \sin_{p,q}(\delta),\ u'(t_0) = b = \alpha \cos_{p,q}(\delta). \tag{3.34}$$

Use of the identity

$$\left|\sin_{p,q}\delta\right|^q + \left|\cos_{p,q}\delta\right|^p = 1$$

shows that

$$\left(\frac{|a|A_{p,q}(|\alpha|,\lambda)}{\alpha}\right)^q + \left(\frac{|b|}{|\alpha|}\right)^p = 1,$$

which gives

$$|\alpha|^p = \frac{\lambda p'}{q}|a|^q + |b|^p. \tag{3.35}$$

Together with (3.34) this enables us to find unique $\alpha \in \mathbb{R}$ and $\delta \in [0,\pi_{p,q})$.

Now that we have solved the initial-value problem, it is a simple matter to handle various boundary-value problems. For the Dirichlet problem we have

Theorem 3.3. *Given $T > 0$, all eigenvalues λ of the Dirichlet problem*

$$\left.\begin{aligned} \left((u')_{(p)}\right)' + \lambda(u)_{(q)} &= 0 \text{ on } (0,T), \\ u(0) = u(T) &= 0, \end{aligned}\right\} \tag{3.36}$$

are of the form

$$\lambda_{n,\alpha} = \left(\frac{n\pi_{p,q}}{T}\right)^q \frac{|\alpha|^{p-q} q}{p'}, \alpha \in \mathbb{R}\backslash\{0\}, n \in \mathbb{N}, \tag{3.37}$$

with corresponding eigenfunctions

$$u_{n,\alpha}(t) = \frac{\alpha T}{n\pi_{p,q}} \sin_{p,q}\left(\frac{n\pi_{p,q}}{T} t\right). \tag{3.38}$$

Proof. Given $\alpha \in \mathbb{R}\backslash\{0\}$, it follows from (3.32) that λ is an eigenvalue of (3.36) if and only if $\sin_{p,q}(A_{p,q}(|\alpha|, \lambda)T) = 0$ This means that $A_{p,q}(|\alpha|, \lambda)T = n\pi_{p,q}$ for some $n \in \mathbb{N}$, from which (3.37) follows. The rest is clear. □

For the Neumann problem similar procedures give

Theorem 3.4. *Given $T > 0$, all eigenvalues of the Neumann problem*

$$\left.\begin{array}{rcl} \left((v')_{(p)}\right)' + \mu (v)_{(q)} & = & 0 \text{ on } (0,T), \\ v'(0) = v'(T) & = & 0, \end{array}\right\} \tag{3.39}$$

are of the form

$$\mu_{n,\alpha} = \left(\frac{n\pi_{p,q}}{T}\right)^q \frac{|\alpha|^{p-q} q}{p'}, \alpha \in \mathbb{R}\backslash\{0\}, n \in \mathbb{N}_0. \tag{3.40}$$

The eigenfunctions corresponding to the zero eigenvalue are the non-zero constants; those corresponding to $\mu_{n,\alpha}$ with $n > 0$ are

$$v_{n,\alpha}(t) = \frac{\alpha T}{n\pi_{p,q}} \sin_{p,q}\left(\frac{n\pi_{p,q}}{T}\left(t - \frac{T}{2n}\right)\right). \tag{3.41}$$

Finally, for the periodic case we have

Theorem 3.5. *Given $T > 0$, all eigenvalues of the periodic problem*

$$\left.\begin{array}{rcl} \left((w')_{(p)}\right)' + \mu (w)_{(q)} & = & 0 \text{ on } (0,T), \\ w(0) & = & w(T) \end{array}\right\} \tag{3.42}$$

are given by (3.40). To the zero eigenvalue corresponds the non-zero constant eigenvectors, while the eigenvectors corresponding to $\mu_{n,\alpha}$ with $n > 0$ are

$$v_{n,\alpha}(t) = \frac{\alpha T}{n\pi_{p,q}} \sin_{p,q}\left(\frac{n\pi_{p,q}}{T}(t - t_n)\right), \tag{3.43}$$

with arbitrary $t_n \in \mathbb{R}$.

Other combinations of boundary conditions can be handled by obvious adaptations of these procedures.

3.4 Eigenfunctions of the p-Laplacian, $n > 1$

The one-dimensional analysis of the last section may be partially carried over to higher dimensions when the underlying space domain is a box. To emphasis the connection with Sect. 3.3 we write the form of the p-Laplacian that we consider as

$$\Delta_p u = \sum_{j=1}^{n} D_j\left((D_j u)_{(p)}\right), \text{ where } (v)_{(p)} = |v|^{p-2} v \text{ and } D_j = \partial/\partial x_j,$$

and the eigenvalue problem we shall study is of the form

$$\Delta_p u = -\lambda (u)_{(p)} \text{ in } \Omega = \prod_{j=1}^{n} (a_j, b_j),$$

with Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega.$$

However, for simplicity of presentation we shall deal with the following case of this problem, in which $n = 2$, as it will illustrate the procedure sufficiently well.

Theorem 3.6. *Let* $1 < p < \infty$. *The Dirichlet problem*

$$\sum_{j=1}^{2} D_j\left((D_j u)_{(p)}\right) = -\lambda (u)_{(p)} \textit{ in } Q := (0,a) \times (0,b),\ u = 0 \textit{ on } \partial Q, \tag{3.44}$$

has eigenvalues

$$\pi_p^p\left((j/a)^p + (k/b)^p\right)p/p' \quad (j,k \in \mathbb{N})$$

and corresponding eigenfunctions that are multiples of

$$\sin_p(j\pi_p x_1/a)\sin_p(k\pi_p x_2/b).$$

Proof. We look for solutions of (3.44) of the form $u(x_1,x_2) = u_1(x_1)u_2(x_2)$ and see that u_1, u_2 must satisfy the equation

$$D_1((u_2 D_1 u_1)_{(p)}) + D_2((u_1 D_2 u_2)_{(p)}) + \lambda (u_1 u_2)_{(p)} = 0,$$

so that

$$(u_2)_{(p)} D_1((D_1 u_1)_{(p)}) + (u_1)_{(p)} D_2((D_2 u_2)_{(p)}) + \lambda (u_1)_{(p)} (u_2)_{(p)} = 0.$$

Division by $(u_1)_{(p)}(u_2)_{(p)}$ shows that

$$\frac{D_1((D_1u_1)_{(p)})}{(u_1)_{(p)}}+\frac{D_2((D_2u_2)_{(p)})}{(u_2)_{(p)}}=-\lambda,$$

from which it follows that for some constant μ,

$$\frac{D_1((D_1u_1)_{(p)})}{(u_1)_{(p)}}=-\mu,\frac{D_2((D_2u_2)_{(p)})}{(u_2)_{(p)}}=-(\lambda-\mu).$$

The boundary condition $u=0$ on ∂Q requires that

$$u_1(0)=u_1(a)=u_2(0)=u_2(b)=0.$$

Hence the original Dirichlet problem decouples into the two problems

$$D_1((D_1u_1)_{(p)})+\mu(u_1)_{(p)}=0 \text{ on } (0,a),\ u_1(0)=u_1(a)=0 \tag{3.45}$$

and

$$D_2((D_2u_2)_{(p)})+(\lambda-\mu)(u_2)_{(p)}=0 \text{ on } (0,b),\ u_2(0)=u_2(b)=0. \tag{3.46}$$

By Theorem 3.3, all solutions of (3.45) are of the form

$$u_1(x_1)=\frac{\alpha a}{j\pi_p}\sin_p\left(\frac{j\pi_p}{a}x_1\right)\ (\alpha\in\mathbb{R},\ j\in\mathbb{N}),\text{ corresponding to}$$
$$\mu=\left(\frac{j\pi_p}{a}\right)^p\frac{p}{p'},$$

and all solutions of (3.46) are of the form

$$u_2(x_2)=\frac{\beta b}{k\pi_p}\sin_p\left(\frac{k\pi_p}{b}x_2\right)\ (\beta\in\mathbb{R},\ j\in\mathbb{N}),\text{ corresponding to}$$
$$\lambda-\mu=\left(\frac{k\pi_p}{b}\right)^p\frac{p}{p'}.$$

Thus λ must be of the form

$$\lambda=\pi_p^p((j/a)^p+(k/b)^p)p/p',$$

and the result follows. □

This theorem gives all the solutions of the Dirichlet eigenvalue problem that are of product form. When $p=2$, all solutions of this problem are of this form, but it is not known whether or not this is true when $p \neq 2$. The Neumann problem may be handled in the same way.

Notes

Note 3.1. The literature on the p-Laplacian and operators that resemble it in some sense is enormous. Here we mention only a few works, beyond those already cited, that seem of particular relevance to our approach. Of special interest is the excellent survey paper by Lindqvist [91]; see also the book [38]. In [10] a Sturm–Liouville theory is developed for the one-dimensional p-Laplacian, following on from the work of [57]; see also [103]. The series of papers by Bognár [12–14] is concerned with the extension of properties known for the one-dimensional p-Laplacian to higher dimensions. She uses the same form of the higher-dimensional p-Laplacian as we do: this lends itself more easily to the separation of variables technique than does the other standard form of the p-Laplacian given by $\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ which, however, has the advantage of invariance under rotations. However, the abstract theory given in Sect. 1.3 can be applied to this other form of the p-Laplacian. All that has to be done is to take $X = \overset{0}{W}{}^1_p(\Omega)$ as before, but this time give it the (equivalent) norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p},$$

where $|\nabla u| = \left(\sum_{j=1}^n |D_j u|^2\right)^{1/2}$. This space X is a closed subspace of the space $L_p(l_2)$, in the notation of Remark 1.1 (ii), and so is uniformly convex, by that Remark. Moreover, the duality map on X corresponding to the gauge function $\mu(t) = t^{p-1}$ can easily be verified to be

$$u \longmapsto -div\left(|\nabla u|^{p-2}\nabla u\right),$$

and now the whole theory follows as before. In particular, we have the existence of a sequence (u_k) of k-weak eigenvectors of the problem

$$-div\left(|\nabla u|^{p-2}\nabla u\right) = \lambda^{-p}|u|^{p-2}u, \; u = 0 \text{ on } \partial\Omega,$$

with a corresponding sequence (λ_k^{-p}) of eigenvalues such that $\lambda_k^{-p} = 0(k^{p/n})$ as $k \to \infty$. This upper estimate of the growth of the eigenvalues is exactly that obtained for the Lyusternik–Schnirelmann eigenvalues in [64] and [66], where lower bounds of the same order are also established. We emphasise that while the first eigenvalue obtained by our method coincides with the first one found by the Lyusternik–Schnirelmann procedure, it is not clear what connection (if any) there is between the higher eigenvalues found by the two procedures, nor whether there are eigenvalues not found by either method.

In [5] an interesting and geometrical approach is followed when studying a class of elliptic operators, including the p-Laplacian, in arbitrary dimensions.

Chapter 4
Hardy Operators

The maps T we shall consider in this chapter act between Lebesgue spaces on an interval (a,b), where b may be infinite, and are of the form

$$Tf(x) = v(x)\int_a^x u(t)f(t)dt,$$

u and v being prescribed functions. They are commonly called Hardy operators, or operators of Hardy type, the operator originally studied by Hardy being that in which $a = 0, b = \infty$ and $v = u = 1$. Necessary and sufficient conditions for the boundedness or compactness of T are given. When u and v are both identically equal to 1 and b is finite, the exact value of the norm of T is determined; it is shown that it is attained at a function expressible in terms of generalised trigonometric functions.

4.1 Conditions for Boundedness and Compactness

Throughout this section we shall assume that $-\infty < a < b \leq \infty$ and $p,q \in [1,\infty]$, while u and v are given real-valued functions such that for all $X \in (a,b)$,

$$u \in L_{p'}(a,X), \tag{4.1}$$

and

$$v \in L_q(X,b), \tag{4.2}$$

where $1/p' = 1 - 1/p$. We denote by $\|\cdot\|_{p,I}$ the usual norm on $L_p(I)$, often writing simply $\|\cdot\|_p$ when $I = (a,b)$. The map T of Hardy type that we shall study is given, as above, by

$$Tf(x) = v(x)\int_a^x u(t)f(t)dt, \quad x \in (a,b), \tag{4.3}$$

and it is well known that the criteria for its boundedness as a map from $L_p(a,b)$ to $L_q(a,b)$ are very different for the cases $p \leq q$ and $p > q$. In the first of these cases the basic result is the following.

J. Lang and D. Edmunds, *Eigenvalues, Embeddings and Generalised Trigonometric Functions*, Lecture Notes in Mathematics 2016, DOI 10.1007/978-3-642-18429-1_4,

Theorem 4.1. *Let $1 \le p \le q \le \infty$ and suppose that (4.1) and (4.2) hold. Then T is a bounded linear map from $L_p(a,b)$ to $L_q(a,b)$, with norm $\|T : L_p(a,b) \to L_q(a,b)\| = \|T\|$, if and only if*

$$A := \sup_{a<X<b} \left\{ \|u\|_{p',(a,X)} \|v\|_{q,(X,b)} \right\} < \infty. \tag{4.4}$$

If (4.4) holds, then

$$A \le \|T\| \le 4A. \tag{4.5}$$

Remark 4.1. A proof of this well known result is given in [42], Chap. 2, where some historical remarks are also provided. When $1 < p \le q < \infty$, Opic (see [102], Comment 3.6, p. 27) has shown that the constant $4A$ on the right-hand side of (4.5) may be replaced by

$$(1+q/p')^{1/q}(1+p'/q)^{1/p'}A, \tag{4.6}$$

which, when $p = q \in (1,\infty)$, gives the best possible constant $p^{1/p}(p')^{1/p'}A$. When $1 < p < q < \infty$, $(a,b) = (0,\infty)$ and $u \notin L_{p'}(0,\infty)$, further improvement is given by Manakov [95] and Read [111], who showed independently that instead of $4A$ one may take

$$\left\{ \frac{\Gamma(q/r)}{\Gamma(1+1/r)\Gamma((q-1)/r)} \right\}^{r/q} A, \; r = \frac{q}{p} - 1. \tag{4.7}$$

Moreover, if either $p = 1$ and $q \in [1,\infty]$, or $q = \infty$ and $p \in (1,\infty]$, it is known that (see [97] and [102], Lemma 5.4)

$$\|T\| = A. \tag{4.8}$$

The first to deal with the case $p > q$ was Maz'ya [97], who established the following result.

Theorem 4.2. *Let $1 \le q < p \le \infty$, $1/s = 1/q - 1/p$ and suppose that (4.1) and (4.2) hold. Then T is a bounded linear map from $L_p(a,b)$ to $L_q(a,b)$ if and only if*

$$B := \left\{ \int_a^b \left(\left(\int_x^b |v(t)|^q \, dt \right)^{1/q} \left(\int_a^x |u(t)|^{p'} \, dt \right)^{1/q'} \right)^s |u(x)|^{p'} \, dx \right\}^{1/s} < \infty, \tag{4.9}$$

in which case

$$q^{1/q}(p'q/s)^{1/q'} B \le \|T\| \le q^{1/q}(p')^{1/q'} B. \tag{4.10}$$

Another version of this characterisation is given in [42], Theorem 2.2.4. To explain this, let $[a,b] = \cup_{i \in \mathscr{I}} \overline{B}_i$, where each $B_i = (a_i, a_{i+1})$ is non-empty and the index set $\mathscr{I}$ may be finite or infinite; denote the family of all such decompositions of $[a,b]$ by $\mathscr{C}$.

Theorem 4.3. *Let $1 \leq q < p \leq \infty$, $1/s = 1/q - 1/p$ and suppose that (4.1) and (4.2) hold. Then T is a bounded linear map from $L_p(a,b)$ to $L_q(a,b)$ if and only if*

$$C := \sup_{\mathscr{C}} \|\{\beta_i\} \mid l_s(\mathscr{I})\| < \infty, \tag{4.11}$$

where $\{\beta_i\}$ is the sequence given by

$$\beta_i = \|u\|_{p',B_{i-1}} \|v\|_{q,B_i}, \; i \in \mathscr{I}, \tag{4.12}$$

corresponding to a decomposition $\{B_i\}_{i\in\mathscr{I}} \in \mathscr{C}$ and $l_s(\mathscr{I})$ is the usual sequence space. If (4.11) holds, then

$$C \leq \|T\| \leq 4C. \tag{4.13}$$

Turning now to characterisations of the compactness of T, we see that there is an even more remarkable difference between the cases $p \leq q$ and $p > q$ than for boundedness. For the first of these we have the following result, the proof of which is given in [102], Theorems 7.3 and 75, and in [42], Theorem 2.3.1.

Theorem 4.4. *Let $1 \leq p \leq q < \infty$ or $1 < p \leq q = \infty$ and suppose that (4.1) and (4.2) hold; put*

$$A(c,d) = \sup_{c<X<d} \left\{ \|u\|_{p',(c,X)} \|v\|_{q,(X,d)} \right\}, a \leq c < d \leq b. \tag{4.14}$$

Then if T is a bounded linear map from $L_p(a,b)$ to $L_q(a,b)$, T is compact if and only if

$$\lim_{c\to a+} A(a,c) = \lim_{d\to b-} A(d,b) = 0. \tag{4.15}$$

Remark 4.2. It will be seen that this result does not cover the case in which $p = 1$ and $q = \infty$. There is a good reason for this, since $T : L_1(a,b) \to L_\infty(a,b)$ $(T \neq 0)$ is never compact. For this, see [45], Remark (b) after Theorem 4.

When $p > q$, the striking result given next, proved in [102], Theorem 7.5 (see also [42], Theorem 2.3.4) asserts that boundedness of T is equivalent to compactness.

Theorem 4.5. *Let $1 \leq q < p \leq \infty$, $1/s = 1/q - 1/p$ and suppose that (4.1) and (4.2) hold. Then T is a compact map from $L_p(a,b)$ to $L_q(a,b)$ if and only if it is bounded.*

4.2 The Norm of the Hardy Operator

Throughout this section we suppose that $a,b \in \mathbb{R}$, with $a < b$, and that $p,q \in [1,\infty]$. We focus on the particular form of the Hardy operator given by the map $H_c : L_p(a,b) \to L_q(a,b)$, where

$$H_c f(x) = \int_c^x f(t)dt, \; c \in [a,b]. \tag{4.16}$$

Theorems 4.1 and 4.2 show that H_c is bounded, and by a simple scaling argument we have

$$\begin{aligned}\left\|H_a : L_p(a,b) \to L_q(a,b)\right\| &= \left\|H_b : L_p(a,b) \to L_q(a,b)\right\| \\ &= (b-a)^{1-\frac{1}{p}+\frac{1}{q}} \left\|H_0 : L_p(0,1) \to L_q(0,1)\right\|. \quad (4.17)\end{aligned}$$

For simplicity we write

$$\gamma_{p,q} = \left\|H_0 : L_p(0,1) \to L_q(0,1)\right\|. \quad (4.18)$$

We propose to calculate $\gamma_{p,q}$ and to determine, where possible, the extremal functions, that is, functions $f \in L_p(0,1)\backslash\{0\}$ such that $\|H_0 f\|_q = \|f\|_p$.

Lemma 4.1. *Suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. Then there exists a non-negative function $f \in L_p(0,1)$, with unit norm, such that $\left\|H_0 : L_p(0,1) \to L_q(0,1)\right\| = \|H_0 f\|$, and any extremal is a multiple of a non-negative extremal.*

Proof. By Theorems 4.4 and 4.5, H_0 is compact. The existence of an extremal $g \in L_p(0,1)$ with unit norm now follows from Proposition 1.10. Put $f = |g|$: since $\|f\|_p = \|g\|_p$ and $\left|\int_0^x g(t)dt\right| \leq \int_0^x f(t)dt$ for all $x \in (0,1)$, the function f is an extremal. Then $\|H_0 f\|_q = \|H_0 g\|_q$, so that $\int_0^x f(t)dt = \left|\int_0^x g(t)dt\right|$ for all $x \in (0,1)$. Thus $\left|\int_0^1 g(t)dt\right| = \int_0^1 |g(t)|\,dt$, which implies that for some $\theta \in \mathbb{R}$, $g = e^{i\theta} f$ (see, for example, [72], Chap. II, Note 12.29). □

Note that it follows from this that $\|H_0\|$ is the same for complex spaces as for real ones.

Theorem 4.6. *Let $p,q \in (1,\infty)$ and let I be the interval $(0,1)$. Then*

$$\left\|H_0 : L_p(I) \to L_q(I)\right\| = (p'+q)^{1-\frac{1}{p'}-\frac{1}{q}} (p')^{1/q} q^{1/p'} / B(1/p', 1/q). \quad (4.19)$$

The extremals are the non-zero multiples of $\cos_{p,q}(\pi_{p,q} x/2)$.

Proof. Let $X = L_p(I)$, $Y = L_q(I)$ and note that X, Y, X^* and Y^* are all uniformly convex. Let J_X, J_Y be the duality maps on X, Y with gauge functions $\mu_X(t) = t^{p-1}$, $\mu_Y(t) = t^{q-1}$ respectively; thus $J_X(f) = |f|^{p-2} f$ and $J_Y(g) = |g|^{q-2} g$ for all $f \in X$ and all $g \in Y$. By Lemma 4.1 and Proposition 1.11, there is a non-negative extremal function $f \in X$, with unit norm, such that

$$H_0^* J_Y H_0 f = \|H_0\|^q J_X f. \quad (4.20)$$

Put $F(x) = H_0 f(x) = \int_0^x f(t)dt$, so that $F' = f$ a.e. Then from (4.20) we have

$$\int_x^1 F^{q-1}(t)dt = \|H_0\|^q (F')^{p-1}(x). \tag{4.21}$$

This implies that $F \in C(I)$ and that

$$F^{q-1}(x) + \|H_0\|^q \left((F')^{p-1}\right)'(x) = 0, \ x \in I, \tag{4.22}$$

so that $(F')^{p-1} \in C^1(I)$. Now let $g \in L_p(I)$ and put $G = H_0 g : G' = g$. While $G(0) = 0$ no matter how g is chosen in $L_p(I)$, we may and shall choose g so that $G(1) \neq 0$. Then from (4.20) we see that

$$\langle G, J_Y H_0 f\rangle_Y = \|H_0\|^q \left\langle G', J_X f\right\rangle_X,$$

and hence

$$\begin{aligned} 0 &= \int_0^1 G(t)F^{q-1}(t)dt - \|H_0\|^q \int_0^1 G'(t)(F')^{p-1}(t)dt \\ &= \int_0^1 G(t)\{F^{q-1}(t) + \|H_0\|^q \left((F')^{p-1}\right)'(t)\}dt - \|H_0\|^q G(1)(F')^{p-1}(1). \end{aligned} \tag{4.23}$$

In view of (4.22), we conclude that $f(1) = F'(1) = 0$.

Since

$$((F')^p)' = ((F')^{p-1}F')' = ((F')^{p-1})'F' + (F')^{p-1}F''$$

and

$$(F')^{p-1}F'' = ((F')^p)'/p,$$

we have

$$((F')^{p-1})'F' = ((F')^p)'/p'.$$

Use of this in (4.22) leads to an equation of p-Laplacian type, and we could proceed from this point onwards by use of the results developed in Sect. 3.3. However, we prefer to give a locally self-contained account and so multiply (4.22) by F' and integrate: with $R = FC^{1/q}$ and $C = F^q(1)$ we obtain

$$R^q(x) + q\|H_0\|^q C^{p/q-1}(R')^p(x)/p' = 1. \tag{4.24}$$

If $R(y) = 1$ for some $y \in (0,1)$, then since R is monotonic increasing and $R(1) = 1$, we must have $R(t) = 1$ and $R'(t) = 0$ for all $t \in (y,1)$. But then from (4.23), with the further specialisation of g so that G is non-negative and zero in $[0,y]$, we see that

$$\int_y^1 G(t)F^{q-1}(t)dt = 0,$$

which is impossible. Hence R cannot attain the value 1 at any point $y < 1$, which implies that $x = 1$ is the first positive zero of R' and hence of f. Thus on $(0,1)$ we have from (4.24),

$$\frac{R'}{(1-R^q)^{1/p}} = \left(\frac{p'}{q\,\|H_0\|^q C^{p/q-1}}\right)^{1/p} = a, \text{ say.} \tag{4.25}$$

By integration we obtain

$$R(x) = \sin_{p,q}(ax),$$

since $R(0) = 0$: thus

$$F(x) = C^{1/q}\sin_{p,q}(ax)$$

and

$$f(x) = F'(x) = aC^{1/q}\cos_{p,q}(ax).$$

Hence a is the first positive zero of $\cos_{p,q}$, so that $a = \pi_{p,q}/2$.

Since $\|f\|_p = 1$, it follows from (2.23) that

$$aC^{1/q}\left(\frac{q}{p'+q}\right)^{1/p} = 1,$$

and so

$$C^{-1/q} = \frac{\pi_{p,q}}{2}\left(\frac{q}{p'+q}\right)^{1/p}. \tag{4.26}$$

To complete the proof, all that is necessary is to substitute the expressions for C and $\pi_{p,q}$ given by (4.26) and (2.13) respectively in the formula for a provided by (4.25). □

We define the following constants:

$$\gamma_{p.q} = \frac{(p'+q)^{1-\frac{1}{p'}-\frac{1}{q}}(p')^{1/q}q^{1/p'}}{2B(1/p',1/q)} \tag{4.27}$$

and

$$\gamma_p = \gamma_{p.p} = (p')^{1/p}p^{1/p'}\pi^{-1}\sin(\pi/p)/2. \tag{4.28}$$

Remark 4.3. The cases in which $p,q \in \{1,\infty\}$ are not covered by this theorem and must be handled by different methods: see, for example, [8]. The conclusions are as follows:

(i) $p \in [1,\infty]$, $q = \infty$: $\|H_0\| = 1$; if $p > 1$, the extremals are all the non-zero constants; if $p = 1$, any non-zero multiple of a non-zero positive measure is an extremal.

(ii) $p = \infty$, $q \in [1,\infty)$: $\|H_0\| = (1+q)^{-1/q}$; the extremals are all the non-zero constants.

(iii) $p \in (1, \infty]$, $q = 1 : \|H_0\| = (1 + p')^{-1/p'}$; the extremals are all the non-zero multiples of $(1 - x)^{1/(p-1)}$.

(iv) $p = 1, q \in [1, \infty) : \|H_0\| = 1$; the extremals are all the non-zero multiples of the Dirac measure at the origin.

Notes

Note 4.1. The formula for $\left\|H_0 : L_p(I) \to L_q(I)\right\|$ provided in Theorem 4.6 was, as far as we can tell, first given in 1940 by Erhard Schmidt [113] in an apparently little-known paper. However, even earlier (in 1938, and unknown to Schmidt) V. Levin [86] had given this formula for the case $p = q$. Unaware of either of these papers, the formula was rediscovered much later, and independently, in [46] (for $p = q$), [8] and [39]. The proof that we give is essentially that of [8]. We refer to [18] and [116, 117] for determination of the best constants for integral operators with positive kernels.

Chapter 5
s-Numbers and Generalised Trigonometric Functions

This chapter gives a review of the various s-numbers and n-widths, their properties and relations; an important subclass of s-numbers, the strict s-numbers, is identified. Then we focus on the Hardy operator (with functions u and v both identically equal to 1) and certain first-order Sobolev embeddings, and show that the generalised trigonometric functions play an essential rôle in the derivation of estimates of s-numbers of these maps. In particular, for the Hardy operator $T : L_p(I) \to L_p(I)$, where $1 < p < \infty$, I is a bounded interval in $\mathbb{R}$ and

$$Tf(x) = \int_a^x f(t)dt,$$

it is shown that all the strict s-numbers of T coincide and are given by an explicit formula.

5.1 s-Numbers and n-Widths

Throughout this section X and Y will stand for Banach spaces, B_X for the closed unit ball in X and I_X for the identity map of X to itself; $B(X,Y)$ will denote the space of all bounded linear maps of X to Y, and we shall write $B(X)$ instead of $B(X,X)$. Given a closed linear subspace M of X, the embedding map of M into X will be denoted by J_M^X and the canonical map of X onto the quotient space $X \backslash M$ by Q_M^X. A subset A of X is said to be *centrally symmetric* if $0 \in A$ and whenever $x \in A$ also $-x \in A$.

We now introduce various so-called n-widths that are important in approximation theory.

Definition 5.1. Let A be a centrally symmetric subset of X and let $n \in \mathbb{N}$.

(i) The *linear n-width* of A with respect to X is defined to be

$$\delta_n(A,X) = \inf_{P_n} \sup_{x \in A} \|x - P_n(x)\|_X,$$

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where the infimum is taken over all $P_n \in B(X,Y)$ with rank n (that is, $\dim P_n(X) = n$). If $P_n^{\delta} \in B(X,Y)$ has the properties that rank $P_n^{\delta} \le n$ and

$$\delta_n(A,X) = \sup_{x\in A} \left\| x - P_n^{\delta}(x) \right\|_X ,$$

it is called an *optimal* linear operator for $\delta_n(A,X)$.

(ii) The *Kolmogorov n-width* of A with respect to X is given by

$$\widetilde{d}_n(A,X) = \inf_{X_n} \sup_{x\in A} \inf_{y\in X_n} \|x-y\|_X ,$$

where the infimum is taken over all n-dimensional subspaces X_n of X. Any subspace X_n^K of X, with dimension at most n, for which

$$\widetilde{d}_n(A,X) = \sup_{x\in A} \inf_{y\in X_n^K} \|x-y\|_X ,$$

is said to be an *optimal* subspace for $\widetilde{d}_n(A,X)$.

(iii) The *Gelfand n-width* of A with respect to X is defined by

$$\widetilde{c}_n(A,X) = \inf_{L_n} \sup_{x\in A\cap L_n} \|x\|_X ,$$

where the infimum is taken over all closed subspaces L_n of X of codimension at most n. Any subspace L_n^G of X, with codimension at most n, for which

$$\widetilde{c}_n(A,X) = \sup\left\{ \|x\|_X : x \in A \cap L_n^G \right\},$$

is called an *optimal* subspace for $\widetilde{c}_n(A,X)$.

(iv) The *Bernstein n-width* of A with respect to X is given by

$$\widetilde{b}_n(A,X) = \sup_{X_{n+1}} \sup\left\{ \lambda \ge 0 : X_{n+1} \cap (\lambda B_X) \subset A \right\},$$

where the outer supremum is taken over all subspaces X_{n+1} of X with dimension $n+1$. Any subspace X_{n+1}^B of X, with dimension $n+1$, for which $X_{n+1}^B \cap \left(\widetilde{b}_n(A,X)B_X\right) \subset A$, is said to be an *optimal* subspace for $\widetilde{b}_n(A,X)$.

The next lemmas provide some basic properties of these widths: the first lemma needs no proof.

Lemma 5.1. *Let A be a centrally symmetric subset of X, let $n \in \mathbb{N}$ and let s_n stand for any of $\delta_n(A,X)$, $\widetilde{d}_n(A,X)$, $\widetilde{c}_n(A,X)$ and $\widetilde{b}_n(A,X)$. Then*

$$\sup_{a\in A} \|a\|_X \ge s_n(A,X) \ge s_{n+1}(A,X),$$

and, if Z is a Banach space such that $A \subset Z \subset X$,

$$s_n(A,X) \leq s_n(A,Z).$$

Lemma 5.2. *Let A be a centrally symmetric subset of X. Then A is relatively compact if and only if A is bounded and $\widetilde{d_n}(A,X) \downarrow 0$ as $n \uparrow \infty$.*

Proof. First suppose that $\overline{A}$ is compact. Then for every $\varepsilon > 0$ there is a finite ε-net of A, that is, a set of points $\{x_1,\ldots,x_n\}$ such that for every $x \in A$, $\min\{\|x-x_i\| : i = 1,\ldots,n\} \leq \varepsilon$. From the definition of $\widetilde{d_n}(A,X)$, considering $X_n := \text{sp }\{x_1,\ldots,x_n\}$, we see that $\widetilde{d_n}(A,X) \leq \varepsilon$; and since $\widetilde{d_{n+1}}(A,X) \leq \widetilde{d_n}(A,X)$, by the last Lemma, it follows that $\widetilde{d_n}(A,X) \downarrow 0$.

For the reverse implication, suppose that A is bounded and $\widetilde{d_n}(A,X) \downarrow 0$ as $n \uparrow \infty$. Since A is bounded, $\widetilde{d_0}(A,X) := \sup\{\|x\|_X : x \in A\} < \infty$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\widetilde{d_n}(A,X) < \varepsilon$ if $n \geq N$, and hence there is an n-dimensional subspace X_n of X with the property that

$$\sup_{a\in A} \inf_{y\in X_n} \|x-y\|_X < \varepsilon.$$

Thus to each $x \in A$ there is a $y \in X_n$ with $\|x-y\|_X < \varepsilon$, so that $\|y\|_X < \widetilde{d_0}(A,X)+\varepsilon$. Since $\{y \in X_n : \|y\|_X \leq \widetilde{d_0}(A,X)+\varepsilon\}$ is a compact subset of X_n, it has an ε-net, which is plainly a 2ε-net for A. The proof is complete. $\square$

When the set A is replaced in Definition 5.1 by the image of the unit ball of a linear map $T : X \to Y$, the n-widths of T are obtained.

Definition 5.2. Let $T : X \to Y$ be linear and bounded; let $n \in \mathbb{N}$. The nth linear width of T is

$$\widetilde{\delta_n}(T) := \widetilde{\delta_n}(T(B_X),Y).$$

The nth Kolmogorov, Gelfand and Bernstein n-widths of T ($\widetilde{d_n}(T), \widetilde{c_n}(T)$ and $\widetilde{b_n}(T)$, respectively) are defined in the obvious analogous manner.

More about n-widths can be found in [28] or [109].

Now we introduce s-numbers. Let $s : T \longmapsto (s_n(T))$ be a rule that attaches to every bounded linear operator acting between any pair of Banach spaces a sequence of non-negative numbers that has the following properties:

- **(S1)** $\|T\| = s_1(T) \geq s_2(T) \geq \ldots 0$.
- **(S2)** $s_n(S+T) \leq s_n(S) + \|T\|$ for $S,T \in B(X,Y)$ and $n \in \mathbb{N}$.
- **(S3)** $s_n(BTA) \leq \|B\|\, s_n(T)\, \|A\|$ whenever $A \in B(X_0,X), T \in B(X,Y), B \in B(Y,Y_0\,)$ and $n \in \mathbb{N}$.
- **(S4)** $s_n(Id : l_2^n \to l_2^n) = 1$ for $n \in \mathbb{N}$.
- **(S5)** $s_n(T) = 0$ when rank $(T) < n$.

We shall call $s_n(T)$ (or $s_n(T : X \to Y)$) the nth s-number of T.
When **(S4)** is replaced by

$$\textbf{(S6)} s_n(Id : E \to E) = 1 \text{ for every Banach space E with} \dim(E) \geq n,$$

we say that $s_n(T)$ is the nth s-number of T in the "strict" sense. Obviously **(S6)** implies **(S4)**, and so for a given operator T the class of s-numbers is larger than that of "strict" s-numbers. More information about s-numbers, and those that are "strict", can be found in [105].

Now we introduce certain important s-numbers and give some relations between them. First, moduli of injectivity and surjectivity are defined and their basic properties established.

Definition 5.3. Let $T \in B(X,Y)$. The modulus of injectivity of T is

$$j(T) := \sup\{\rho \geq 0 : \|Tx\|_Y \geq \rho \|x\|_X \text{ for all } x \in X\};$$

the modulus of surjectivity of T is

$$q(T) := \sup\{\rho \geq 0 : T(B_X) \supset \rho B_Y\}.$$

Lemma 5.3. *Let Z be a Banach space.*

(i) If $S,T \in B(X,Y)$, then

$$j(S+T) \leq j(S) + \|T\| \text{ and } q(S+T) \leq q(S) + \|T\|.$$

(ii) If $T \in B(X,Z)$ and $S \in B(Z,Y)$, then

$$j(ST) \leq \|S\| \, j(T) \text{ and } q(ST) \leq q(S) \|T\|.$$

Moreover, if T is surjective, then

$$j(ST) \leq j(S) \|T\|,$$

while if S is surjective, then

$$q(ST) \leq \|S\| q(T).$$

Proof. The assertions regarding j are obvious. For q we prove only (i). Suppose that $q(S+T) > \|T\|$ and let $0 < \varepsilon < q(S+T) - \|T\|$. Put $\rho = q(S+T) - \varepsilon$, take $y \in B_Y$ and define a sequence $\{x_i\}$ of elements of X by induction:

$$Sx_1 + Tx_1 = (\rho - \|T\|)y, \ \|x_1\|_X \leq (\rho - \|T\|)/\rho,$$

$$Sx_{n+1} + Tx_{n+1} = Tx_n, \ \|x_{n+1}\|_X \leq \|Tx_n\|_Y / \rho \ (n \in \mathbb{N}).$$

Clearly

$$\|x_n\|_X \leq \left(\frac{\|T\|}{\rho}\right)^{n-1} \frac{\rho - \|T\|}{\rho} \; (n \in \mathbb{N}).$$

Since $\|T\| < \rho$, it follows that there exists $x \in X$ such that $x = \sum_1^\infty x_n$. Moreover, $\|x\|_X \leq 1$ and $Sx = (\rho - \|T\|)y$: hence $S(B_X) \supset (\rho - \|T\|)B_Y$. This guarantees the following inequality and so concludes the proof:

$$q(S) \geq \rho - \|T\| = q(S+T) - \|T\| - \varepsilon.$$

□

Definition 5.4. Let $T \in B(X,Y)$ and $n \in \mathbb{N}$. Then the nth approximation, isomorphism, Gelfand, Bernstein, Kolmogorov, Mityagin, Weyl, Chang and Hilbert numbers of T are defined by

$$a_n(T) = \inf\{\|T - F\| : F \in B(X,Y), \text{ rank } (T) < n\};$$
$$i_n(T) = \sup\left\{\|A\|^{-1} \|B\|^{-1}\right\},$$

where the supremum is taken over all possible Banach spaces G with dim $(G) \geq n$ and maps $A \in B(Y,G)$, $B \in B(G,X)$ such that ATB is the identity on G;

$$c_n(T) = \inf\left\{\left\|TJ_M^X\right\| : \text{codim}(M) < n\right\};$$
$$b_n(T) = \sup\left\{j\left(TJ_M^X\right) : \dim(M) \geq n\right\};$$
$$d_n(T) = \inf\left\{\left\|Q_N^Y T\right\| : \dim(N) < n\right\};$$
$$m_n(T) = \sup\left\{q\left(Q_N^Y T\right) : \text{codim}(N) \geq n\right\};$$
$$x_n(T) = \sup\{a_n(TA) : \|A : l_2 \to X\| \leq 1\};$$
$$y_n(T) = \sup\{a_n(BT) : \|B : Y \to l_2\| \leq 1\};$$
$$h_n(T) = \sup\{a_n(BTA) : \|A : l_2 \to X\|, \|B : Y \to l_2\| \leq 1\},$$

respectively. Note that the validity of the definition of $i_n(T)$ follows from [105], Lemma 1.1.

The reason for introducing these numbers is given in the next lemma.

Lemma 5.4. *The approximation, isomorphism, Gelfand, Bernstein, Kolmogorov, Mityagin, Weyl, Chang and Hilbert numbers are s-numbers; the first six of these are strict s-numbers.*

Proof. For the approximation numbers we prove only **(S6)**. Suppose there is a Banach space E, with $\dim(E) \geq n$, such that $a_n(Id : E \to E) < 1$. Then there exists $A \in B(E)$ with rank $(A) < n$ and $\|Id - A\| < 1$. Since $A = Id - (Id - A)$ is invertible in $B(E)$ (via the Neumann series), we have rank $(A) \geq n$: contradiction. Thus **(S6)** holds.

All the claimed properties of the isomorphism numbers are clear apart from **(S2)**. Suppose that $i_n(S+T) > \|T\|$ and let $\varepsilon \in (0, i_n(S+T) - \|T\|)$. Then there exist a

Banach space G, with $\dim(G) \geq n$, and maps $A \in B(G,X), B \in B(Y,G)$ such that $I_G = B(S+T)A$ and $\|B\|^{-1} \|A\|^{-1} \geq i_n(S+T) - \varepsilon > \|T\|$. Hence $\|BTA\| < 1$ and so the operator

$$BSA = B(S+T)A - BTA = I_G - BTA$$

is invertible. Since

$$I_G = (I_G - BTA)^{-1} BSA \text{ and } \left\|(I_G - BTA)^{-1}\right\| \leq (1 - \|BTA\|)^{-1},$$

we obtain

$$\begin{aligned} i_n(S) &\geq \left\|(I_G - BTA)^{-1} B\right\|^{-1} \|A\|^{-1} \geq (1 - \|BTA\|) \|B\|^{-1} \|A\|^{-1} \\ &\geq \|B\|^{-1} \|A\|^{-1} - \|T\| \geq i_n(S+T) - \varepsilon - \|T\|. \end{aligned}$$

Thus

$$i_n(S+T) \leq i_n(S) + \|T\| + \varepsilon.$$

That **(S2)** now follows is clear.

The proof of the assertion regarding Gelfand numbers is routine and so is omitted. For the Bernstein numbers we prove only **(S3)**. Suppose that $\varepsilon \in (0, b_n(BTA))$. Then there is a subspace M_0 of X_0, with $\dim(M_0) \geq n$, for which $b_n(BTA) - \varepsilon \leq j\left(BTAJ_{M_0}^{X_0}\right)$. Denote by A_0 the restriction of A to M_0, viewed as a map to $M := A(M_0)$. Then

$$BTAJ_{M_0}^{X_0} = BTJ_M^X A_0 \text{ and } \|A_0\| \leq \|A\|.$$

By Lemma 5.3 (ii) it follows that

$$0 < b_n(BTA) - \varepsilon \leq j\left(BTJ_M^X A_0\right) \leq \left\|BTJ_M^X\right\| j(A_0),$$

which implies that $j(A_0) > 0$. Hence A_0 is injective, so that $\dim(M) \geq n$. Since A_0 is surjective, Lemma 5.3 (ii) gives

$$b_n(BTA) - \varepsilon \leq j\left(BTJ_M^X A_0\right) \leq \|B\| \, j(TJ_M^X) \, \|A_0\| \leq \|B\| \, b_n(T) \, \|A\|,$$

which establishes **(S3)**.

For Mityagin numbers we only show **(S2)**. Set $\varepsilon > 0$: then there exists a subspace N of Y with $\operatorname{codim}(N) \geq n$ such that

$$q(Q_N^Y(S+T)) \geq m_n(S+T) - \varepsilon.$$

Using Lemma 5.3 (i) we have

$$\begin{aligned} m_n(S+T) &\leq q(QN^Y(S+T)) + \varepsilon \leq q(Q_N^Y S) + \|Q_N^Y(T)\| + \varepsilon \\ &\leq m_n(S) + \|T\| + \varepsilon. \end{aligned}$$

We omit proofs of the assertions regarding the remaining numbers as they are either obvious or similar to the proofs given for the other numbers. □

Lemma 5.5. *The approximation numbers are the largest s-numbers.*

Proof. Let $n \in \mathbb{N}$, suppose that $T, A \in B(X,Y)$, with rank $(A) < n$, and let s_n be some s-number. Then

$$s_n(T) \leq s_n(A) + \|T - A\| = \|T - A\|,$$

which means that $s_n(T) \leq a_n(T)$. □

We next introduce two kinds of space that prove to have interesting properties in connection with s-numbers.

Definition 5.5. A Banach space Y is said to have the extension property if, for every map S_0 that maps a subspace of an arbitrary Banach space X into Y, there is a map $S \in B(X,Y)$ such that $\|S\| = \|S_0\|$ and $S_0 x = SJ_M^X x$ for every $x \in M$.

Definition 5.6. A Banach space X has the lifting property if, for every $S_0 \in B(X, Y/N)$, where N is any closed linear subspace of an arbitrary Banach space Y, and any $\varepsilon > 0$, there is a map $S \in B(X,Y)$ such that $\|S\| \leq (1+\varepsilon)\|S_0\|$ and $S_0 x = Q_N^Y S x$ for all $x \in X$.

The significance of these properties becomes clear as a result of the next two theorems.

Theorem 5.1. *If Y has the extension property, then for all $n \in \mathbb{N}$ and all $S \in B(X,Y)$,*

$$c_n(S) = a_n(S).$$

Proof. Let $S \in B(X,Y)$. Since the approximation numbers are the largest s-numbers, it is enough to show that $a_n(S) \leq c_n(S)$. Let $\varepsilon > 0$. There is a subspace M of X, with codim $(M) < n$, such that $\|SJ_M^X\| \leq c_n(S) + \varepsilon$. As Y has the extension property, there is an extension $T \in B(X,Y)$ of SJ_M^X with $\|T\| = \|SJ_M^X\|$. Put $A = S - T$: then rank $(A) < n$ and $Ax = 0$ for every $x \in M$. Hence

$$a_n(S) \leq \|S - A\| = \|T\| = \|SJ_M^X\| \leq c_n(S) + \varepsilon.$$

□

Theorem 5.2. *If X has the lifting property, then for all $n \in \mathbb{N}$ and all $S \in B(X,Y)$,*

$$d_n(S) = a_n(S).$$

Proof. Let $S \in B(X,Y)$. We simply have to show that $a_n(S) \leq d_n(S)$. Let $\varepsilon > 0$ and let N be a subspace of Y such that $\dim(N) < n$ and $\|Q_N^Y S\| \leq d_n(S) + \varepsilon$. Then there is a (lifting) map $T \in B(X,Y)$ for which $\|T\| \leq (1+\varepsilon)\|Q_N^Y S\|$ and $Q_N^Y S x = Q_N^Y T x$

for every $x \in X$. Define $A = S - T$. Then $Ax \in N$ for all $x \in X$: rank $(A) < n$. Hence

$$a_n(S) \leq \|S - A\| = \|T\| \leq (1+\varepsilon)(d_n(S) + \varepsilon),$$

from which the result follows. □

Lemma 5.6. *The isomorphism numbers are the smallest strict s-numbers and the Hilbert numbers are the smallest s-numbers.*

Proof. Let $T \in B(X,Y)$, $B \in B(G,X)$, $A \in B(Y,G)$ be such that $I_G = ATB$ and $\dim(G) \geq n$; let s_n be some strict s-number. Then

$$1 = s_n(I_G) \leq \|A\| \, s_n(T) \, \|B\|,$$

from which we have $i_n(T) \leq s_n(T)$. The proof for the Hilbert numbers is a natural variation of this argument. □

Now the notions of injective and surjective s-numbers are introduced.

Definition 5.7. Let s_n be an s-number. Then

(i) s_n is said to be *injective* if, for every subspace N of Y and every $S \in B(X,N)$,

$$s_n\left(J_N^Y S\right) = s_n(S);$$

(ii) s_n is called *surjective* if, for every quotient space $X \backslash M$ of X and every $S \in B(X \backslash M, Y)$,

$$s_n\left(SQ_M^X\right) = s_n(S).$$

Lemma 5.7. *The Gelfand numbers are the largest injective s-numbers; the Bernstein numbers are the smallest injective strict s-numbers.*

Proof. It is clear that the Gelfand number are injective. Note that every Banach space Y is a subspace of a Banach space Y^{∞} which has the extension property (see [106], C.3.3). Let $S \in B(X,Y)$; then $c_n(S) = c_n\left(J_Y^{\infty} S\right)$, where J_Y^{∞} is the embedding from Y to Y^{∞}. Together with Theorem 5.1 this gives

$$c_n(S) = a_n\left(J_Y^{\infty} S\right). \tag{5.1}$$

Now let s_n be an injective s-number. Then $s_n(S) = s_n\left(J_Y^{\infty} S\right) \leq a_n\left(J_Y^{\infty} S\right)$, the inequality following from Lemma 5.5. In view of (5.1), this concludes the proof for the Gelfand numbers.

To handle the Bernstein numbers, let $T \in B(X,Y)$, s_n be an injective strict s-number, and M be a subspace of X with $\dim(M) \geq n$; suppose $j\left(TJ_M^X\right) > 0$. Put $M_0 = T(M)$ and denote by T_0 the restriction of T to M, viewed as a map from

M to M_0. Then T_0 is invertible and

$$\left\|T_0^{-1}\right\| = j\left(TJ_M^X\right)^{-1}.$$

Hence

$$\begin{aligned} 1 = s_n(I_M) \leq s_n(T_0)\left\|T_0^{-1}\right\| &= s_n(J_{M_0}^Y T_0)\left\|T_0^{-1}\right\| \\ \leq s_n(TJ_M^X)\left\|T_0^{-1}\right\| &\leq s_n(T)\, j\left(TJ_M^X\right)^{-1}. \end{aligned}$$

Hence $j\left(TJ_M^X\right) \leq s_n(T)$, and so $b_n(T) \leq s_n(T)$. □

Lemma 5.8. *The Kolmogorov numbers are the largest surjective s-numbers; the Mityagin numbers are the smallest surjective numbers.*

Proof. That the Kolmogorov numbers and the Mityagin numbers are surjective is plain. Every Banach space X can be identified with a quotient space of some Banach space X^1 with the lifting property (see [106], C.3.7). Let Q_X^1 be the canonical map from X^1 onto X and let $S \in B(X,Y)$. Then

$$d_n(S) = d_n(SQ_X^1) = a_n(SQ_X^1).$$

Let s_n be any surjective s-number. Then since the approximation numbers are the largest s-numbers,

$$s_n(S) = s_n(SQ_X^1) \leq a_n(SQ_X^1) = d_n(S).$$

The proof that the Mityagin numbers are the smallest surjective numbers is a simple modification of the proof that the Bernstein numbers are the smallest injective strict s-numbers given in the proof of Lemma 5.7. □

In the next theorem we give connections between the s-numbers of a map T and those of its dual T'. Proofs of the assertions concerning approximation numbers are given in [74] and [55]; for the other claims see [105] and [107].

Theorem 5.3. *Let $T \in B(X,Y)$ and $n \in \mathbb{N}$. Then*

$$\begin{gathered} a_n(T') \leq a_n(T) \leq 5a_n(T'), \\ c_n(T) = d_n(T') \text{ and } m_n(T) = b_n(T'), \\ y_n(T) = x_n(T') \text{ and } x_n(T) = y_n(T'), \\ h_n(T) = h_n(T') \text{ and } i_n(T) \leq i_n(T'). \end{gathered}$$

If, in addition, T is compact, then

$$a_n(T) = a_n(T') \text{ and } d_n(T) = c_n(T').$$

Combination of the above lemmas and theorems gives

Theorem 5.4. *Let* $T \in B(X,Y)$ *and* $n \in \mathbb{N}$. *Then*

$$a_n(T) \geq c_n(T) \geq b_n(T) \geq i_n(T) \geq h_n(T)$$

and

$$a_n(T) \geq d_n(T) \geq i_n(T) \geq h_n(T).$$

Two further inequalities follow.

Theorem 5.5. *Let* $T \in B(X,Y)$ *and* $n \in \mathbb{N}$. *Then*

$$d_n(T) \geq b_n(T).$$

Proof. From the definition of the Kolmogorov and Bernstein numbers,

$$d_n(T) = \inf\left\{\left\|Q_N^Y T\right\| : \dim(N) < n\right\}, \, b_n(T) = \sup\left\{j\left(TJ_M^X\right) : \dim(M) \geq n\right\},$$

it follows that it is enough to show that $\left\|Q_N^Y T\right\| \geq j\left(TJ_M^X\right)$. Suppose that $j\left(TJ_M^X\right) > 0$. Then $\dim(T(M)) \geq n$ and there exists $x \in M$ such that $\left\|Q_N^Y Tx\right\|_Y = \|Tx\|_Y = 1$. Since $\|Tx\|_Y = j\left(TJ_M^X\right)\|x\|_X$ and $\left\|Q_N^Y Tx\right\|_Y \leq \left\|Q_N^Y T\right\| \|x\|_X$, the inequality follows. □

Theorem 5.6. *Let* $T \in B(X,Y)$ *and* $n \in \mathbb{N}$. *Then*

$$c_n(T) \geq m_n(T).$$

Proof. From Theorems 5.3 and 5.5 we have

$$c_n(T) = d_n(T') \geq b_n(T') = m_n(T).$$

□

Finally, we summarise some of the information given above concerning the various s-numbers.

Theorem 5.7. *Let* $T \in B(X,Y)$. *All the numbers listed in Definition 5.4 are s-numbers;* $a_n(T), i_n(T), c_n(T), d_n(T)$ *and* $b_n(T)$ *are strict s-numbers. The following inequalities hold for every* $n \in \mathbb{N}$:

$$\begin{aligned} a_n(T) &\geq \max(c_n(T), d_n(T)) \geq \min(c_n(T), d_n(T)) \\ &\geq \min(b_n(T), m_n(T)) \geq i_n(T), \end{aligned}$$

and

$$a_n(T) \geq \max(x_n(T), y_n(T)) \geq \min(x_n(T), y_n(T)) \geq h_n(T).$$

The approximation numbers are the largest s-numbers, the Hilbert numbers are the smallest s-numbers and the isomorphism numbers are the smallest strict s-numbers.

Remark 5.1. Note the difference between the Gelfand, Bernstein and Kolmogorov numbers of Definition 5.4 and the corresponding n-widths given by Definition 5.2. For example, $d_n(T) = \widetilde{d}_{n-1}(T)$. We refer to [109], Chap. 2 and [107], 6.2.6 for further details and comments. These books are also excellent sources of information about the properties of s-numbers.

5.2 The *s*-Numbers of Hardy Operators

Let $-\infty < a < b < \infty$, put $I = (a,b)$, let $p,q \in (1,\infty)$ and consider the Hardy integral operator T_c defined, for each $c \in [a,b]$ by

$$T_c f(x) = \int_c^x f(t)dt, \tag{5.2}$$

and regarded as a map from $L_p(I)$ to $L_q(I)$. It is well known that T_c is compact. In this section we study the s-numbers of T_c : we show that when $p = q$, all its strict s-numbers are equal and with the help of generalised trigonometric functions obtain exact values for these numbers.

The quantity A_0 next defined turns out to play a key rôle in the approximation of T_c.

Definition 5.8. Let $J := (c,d) \subset I$. We define

$$A_0(J) = \sup_{\|u\|_{p,J}>0} \inf_{\alpha\in\mathbb{R}} \frac{\|\int_c^{\cdot} u(t)dt - \alpha\|_{p,J}}{\|u\|_{p,J}}.$$

Plainly

$$A_0(J) = \sup_{\|u\|_{p,J}>0} \inf_{c\le y\le d} \frac{\left\|\int_y^{\cdot} u(t)dt\right\|_{p,J}}{\|u\|_{p,J}}.$$

The following lemmas give some basic properties of A_0.

Lemma 5.9. *Let $\{I_n\}$ be a decreasing sequence of subintervals of I with $|I_n| \to 0$ as $n \to \infty$. Then $\{A_0(I_n)\}$ is a decreasing sequence bounded above by $A_0(I)$ and with limit* 0.

Proof. Let $n \in \mathbb{N}$ and suppose that any $u \in L_p(I_{n+1})$ is extended by 0 outside I_{n+1}. Then

$$\begin{aligned} A_0^p(I_{n+1}) &= \sup_{\|u\|_{p,I_{n+1}}>0} \inf_{\alpha\in\mathbb{R}} \frac{\|\int_c^\cdot u(t)dt-\alpha\|_{p,I_{n+1}}^p}{\|u\|_{p,I_{n+1}}^p} \\ &\le \sup_{\|u\|_{p,I_{n+1}}>0} \inf_{\alpha\in\mathbb{R}} \frac{\|\int_c^\cdot u(t)dt-\alpha\|_{p,I_n}^p}{\|u\|_{p,I_n}^p} \\ &\le \sup_{\|u\|_{p,I_n}>0} \inf_{\alpha\in\mathbb{R}} \frac{\|\int_c^\cdot u(t)dt-\alpha\|_{p,I_n}^p}{\|u\|_{p,I_n}^p} = A_0^p(I_n). \end{aligned}$$

Moreover, for any interval $J \subset I$,

$$\begin{aligned} A_0(J) &\le \sup_{\|u\|_{p,J}=1} \left\| \int_c^\cdot u(t)dt \right\|_{p,J} \\ &\sup_{\|u\|_{p,J}=1} \left\| \left(\int_J |u(t)|^p\, dt \right)^{1/p} |J|^{1/p'} \right\|_{p,J} = |J|^{1/p'}. \end{aligned}$$

Hence $A_0(I_n) \to 0$ as $n \to \infty$. □

Lemma 5.10. *Let $(x,y) \subset I$. Then $A_0((x,y))$ is a continuous function of x and y.*

Proof. For simplicity we shall write $A_0(x,y)$ instead of $A_0((x,y))$; we also adopt the same understanding about the extension of functions as in the last proof. Suppose that there are $x,y \in I$ and $\varepsilon > 0$ such that $A_0(x,y+h_n) - A_0(x,y) > \varepsilon$ for some sequence $\{h_n\}$ with $0 < h_n \downarrow 0$ as $n \uparrow \infty$. Then there exists $\varepsilon_1 > 0$ such that $A_0^p(x,y+h_n) - A_0^p(x,y) > \varepsilon_1$ for all $n \in \mathbb{N}$. For economy of expression write

$$I_{w,z} = \inf_{\alpha\in\mathbb{R}} \frac{\|\int_x^\cdot u(s)ds-\alpha\|_{p,(x,w)}^p}{\|u\|_{p,(x,z)}^p}.$$

Then for all $h > 0$ we have

$$\begin{aligned} A_0^p(x,y+h) - A_0^p(x,y) &= \sup_{\|u\|_{p,(x,y+h)}>0} I_{y+h,y+h} - \sup_{\|u\|_{p,(x,y)}>0} I_{y,y} \\ &\le \sup_{\|u\|_{p,(x,y+h)}>0} \left\{ I_{y+h,y+h} - I_{y,y+h} \right\} \\ &\le \sup_{\|u\|_{p,(x,y+h)}>0} \frac{\|\int_x^\cdot u(s)ds\|_{p,(y,y+h)}^p}{\|u\|_{p,(x,y+h)}^p} \\ &\le |(y,y+h)|^{p/p'} = h^{p/p'}, \end{aligned}$$

and we have a contradiction. Hence $A_0(x,y+h) \to A_0(x,y)$ as $h \to 0$. In the same way it can be shown that $A_0(x+h,y) \to A_0(x,y)$ as $h \to 0$. □

Lemma 5.11. *Let $J=(c,d) \subset I$. Then there is a function $f \in L_p(J)$ and a point $s \in [c,d]$ such that*

$$A_0(J) = \frac{\|\int_s^\cdot f(t)dt\|_{p,J}}{\|f\|_{p,J}} = \inf_{\alpha\in\mathbb{R}} \frac{\|\int_c^\cdot f(t)dt - \alpha\|_{p,J}}{\|f\|_{p,J}}.$$

Proof. There is a sequence $\{f_n\}$ of functions in $L_p(J)$, with $\|f_n\|_{p,J} = 1$ for each $n \in \mathbb{N}$, and a sequence of numbers $\{s_n\}$ from $[c,d]$ such that

$$\left\|\int_{s_n}^\cdot f_n(t)dt\right\|_{p,J} + 1/n = \inf_{\alpha\in\mathbb{R}} \left\|\int_c^\cdot f_n(t)dt - \alpha\right\|_{p,J} + 1/n > A_0(J).$$

Since $T_c : L_p(J) \to L_p(J)$ is compact, there is a subsequence of $\{f_n\}$, again denoted by $\{f_n\}$ for convenience, which converges weakly in $L_p(J)$, to f, say. As $T_c : L_p(J) \to L_p(J)$ is compact, T_c also acts compactly from $L_p(J)\backslash sp\{1\}$, the quotient space modulo constants, to itself, where $\|h\|_{L_p(J)\backslash sp\{1\}} := \inf_{\alpha\in\mathbb{R}} \|h-\alpha\|_{p,J}$; moreover, $T_c f_n \to T_c f$ in $L_p(J)\backslash sp\{1\}$. Using the facts that $\|f\|_{p,J} \le \liminf \|f_n\|_{p,J}$ and $\|T_c f\|_{L_p(J)\backslash sp\{1\}} = A_0(J)$, we conclude that $\|f\|_{p,J} = 1$. Because

$$F(u) := \frac{\|\int_u^\cdot f(t)dt\|_{p,J}}{\|f\|_{p,J}}$$

depends continuously on u, there exists $s \in [c,d]$ such that

$$\frac{\|\int_s^\cdot f(t)dt\|_{p,J}}{\|f\|_{p,J}} = \inf_{c\le u\le d} \frac{\|\int_u^\cdot f(t)dt\|_{p,J}}{\|f\|_{p,J}} = A_0(J).$$

Thus f has all the properties required in the theorem. □

Lemma 5.12. *Let $J=(c,d) \subset I$ and suppose that f and s are as in the last lemma. Then f may be chosen so that $s=(c+d)/2$, $f(c+)=f(d-)=0$ and f is odd about $(c+d)/2$.*

Proof. Set $u(x) = \int_s^x f(t)dt$, $u_+(x) = \max\{u(x),0\}$ and $u_-(x) = \min\{u(x),0\}$; then $\|u_+\|_{p,J} = \|u_-\|_{p,J}$, $u = u_+ - u_-$ and $|\{x : u(x)=0\}| = 0$. For any $h \in W_p^1(J)$, $h \ge 0$, we have $\|h'\|_{p,J} \ge \left\|(h^*)'\right\|_{p,(0,|J|)}$, where h^* is the non-increasing rearrangement of h (see, for example, [42], Chap. 3). Define $r = |\{x : u(x) > 0\} \cap J|$ and

$$g(x) = \begin{cases} u_+^*(x-c) & \text{if } c \le x \le c+r, \\ -u_-^*(d-x) & \text{if } c+r \le x \le d. \end{cases}$$

Then

$$\frac{\|g\|_{p,J}^p}{\|g'\|_{p,J}^p} = \frac{\left\|u_+^*\right\|_{p,(0,|J|)}^p + \left\|u_-^*\right\|_{p,(0,|J|)}^p}{\left\|(u_+^*)'\right\|_{p,(0,|J|)}^p + \left\|(u_-^*)'\right\|_{p,(0,|J|)}^p} = \frac{\|u_+\|_{p,J}^p + \|u_-\|_{p,J}^p}{\left\|(u_+^*)'\right\|_{p,(0,|J|)}^p + \left\|(u_-^*)'\right\|_{p,(0,|J|)}^p}$$
$$\geq \frac{\|u\|_{p,J}^p}{\|u_+{}'\|_{p,J}^p + \|u_-{}'\|_{p,J}^p} = \frac{\|u\|_{p,J}^p}{\|u\|_{p,J}^p} = A_0^p(J).$$

Hence

$$\frac{\|g\|_{p,J}}{\|g'\|_{p,J}} = A_0(J),$$

and $\|g_+\|_{p,J} = \|g_-\|_{p,J}$. Moreover, g is monotone and $g(c+r) = 0$, where $c < c+r < d$.

Now we show that $g((c+d)/2) = 0$; that is, $r = (c+d)/2$. Put $J_1 = (c, c+r)$ and $J_2 = (c+r, d)$. Then

$$\frac{\|g\|_{p,J_1}^p + \|g\|_{p,J_2}^p}{\|g'\|_{p,J_1}^p + \|g'\|_{p,J_2}^p} = A_0^p(J). \tag{5.3}$$

Since $A_0(J) = |J| A_0((0,1))$, we see that

$$\frac{\|g\|_{p,J_1}^p}{\|g'\|_{p,J_1}^p} \leq A_0^p((0,1))\,|J_1|^p\,2^p.$$

For if not, then we can define a function h by

$$h(x) = \begin{cases} g(x), & x \in (c, c+r), \\ -g(-x+2(r+c)), & x \in (c+r, c+2r), \end{cases}$$

so that

$$\inf_{\alpha\in\mathbb{R}} \|h-\alpha\|_{p,(c,c+2r)} = \|h\|_{p,(c,c+2r)} \text{ and } \frac{\|h\|_{p,(c,c+2r)}^p}{\|h'\|_{p,(c,c+2r)}^p} > A_0^p((c, c+2r)),$$

in contradiction to the definition of A_0. Similarly we have

$$\frac{\|g\|_{p,J_2}^p}{\|g'\|_{p,J2}^p} \leq A_0^p((0,1))\,|J_2|^p\,2^p.$$

Next, observe that (5.3) holds if and only if

$$\frac{\|g\|_{p,J_1}}{\|g'\|_{p,J_1}} = \frac{\|g\|_{p,J_2}}{\|g'\|_{p,J_2}} = A_0(J)$$

(remember that $\|g\|_{p,J_1} = \|g\|_{p,J_2}$). Hence $|J_1| = |J_2|$, so that $c+r = (c+d)/2$; moreover, we may suppose that g is odd with respect to $(c+d)/2$, that is, $g(x) = -g(-x+c+d)$. Thus $s = (c+d)/2$.

We now show that $g'(c) = g'(d) = 0$. Note that $g(c) = -g(d) \geq 0$. Suppose that $g'(c) = -g'(d) < 0$. Then there are a number $z < 0$ and a sequence of numbers $\{x_n\}_{n=1}^{\infty}$ such that $x_n > c, x_n \to c$ and

$$\frac{g(c) - g(x_n)}{c - x_n} < z < 0,$$

so that $\int_c^{x_n} g'(t)dt < (x_n - c)z$. A similar procedure can be carried out in the neighbourhood of d. Then

$$|z|(x_n - c) < \int_c^{x_n} |g'(t)|\, dt \leq \left(\int_c^{x_n} |g'(t)|^p\, dt\right)^{1/p} (x_n - c)^{1/p'};$$

and

$$A_0^p(J) = \frac{\int_c^{x_n} |g|^p + \int_{x_n}^d |g|^p}{\int_c^{x_n} |g'|^p + \int_{x_n}^d |g'|^p} \leq \frac{(x_n - c)\,|g(c)|^p + \int_{x_n}^d |g|^p}{(x_n - c)\,|z|^p + \int_{x_n}^d |g'|^p}.$$

Since $A_0(J) > 0$ and $|z| > 0$, plainly

$$|g(c)|^p < |z|^p A_0^p(J) + |g(c)|^p$$

and there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$,

$$(x_n - c)\,|g(c)|^p < (x_n - c)\,|z|^p\, \frac{\int_{x_n}^d |g|^p}{\int_{x_n}^d |g'|^p} + (x_n - c)\,|g(c) - z(x_n - c)|^p.$$

Thus

$$\left(\int_{x_n}^d |g|^p\right)\left(\int_{x_n}^d |g'|^p\right) + (x_n - c)\,|g(c)|^p \left(\int_{x_n}^d |g'|^p\right)$$

is less than

$$\begin{aligned}&\left(\int_{x_n}^d |g|^p\right)\left(\int_{x_n}^d |g'|^p\right) + (x_n - c)\,|z|^p \left(\int_{x_n}^d |g|^p\right)\\ &+ (x_n - c)\,|g(c) - z(x_n - c)|^p \left(\int_{x_n}^d |g'|^p\right)\\ &+ |z|^p\,|g(c) - z(x_n - c)|^p\,(x_n - c)^2.\end{aligned}$$

It follows that for all $n > n_1$,

$$\frac{\int_{x_n}^d |g|^p + (x_n - c)\,|g(c)|^p}{\left(\int_{x_n}^d |g'|^p\right) + (x_n - c)\,|z|^p} < \frac{\int_{x_n}^d |g|^p + (x_n - c)\,|g(x_n)|^p}{\left(\int_{x_n}^d |g'|^p\right)}.$$

This means that for $l_n := \chi_{(x_n,d)} g + \chi_{(c,x_n)} g(x_n)$ we have

$$A_0^p(J) < \frac{\int_c^d |l_n|^p}{\int_c^d |l_n'|^p} \text{ for all } n > n_1.$$

In view of the antisymmetry of g we define a function

$$r_n(x) = \chi_{(c,d+c-x_n)} g(x) + \chi_{(d+c-x_n,d)} g(d+c-x_n),$$

and have

$$A_0^p(J) < \frac{\int_c^d |r_n|^p}{\int_c^d |r_n'|^p} \text{ for all } n > n_1.$$

Finally we define $k_n(x) = \chi_{(x_n,d+c-x_n)} g(x) + \chi_{(d+c-x_n,d)} g(d+c-x_n) + \chi_{(c,x_n)} g(x_n)$. Then for large enough n,

$$A_0(J) < \inf_{\alpha\in\mathbb{R}} \frac{\|k_n - \alpha\|_{p,J}}{\|k_n'\|_{p,J}}.$$

As this contradicts the definition of $A_0(J)$ it follows that $g'(c) = g'(d) = 0$. □

Theorem 5.8. *Let $J = (c,d) \subset I$. Then*

$$A_0(J) = \frac{\left\|\int_{(c+d)/2}^{\cdot} u(t)dt\right\|_{p,J}}{\|u\|_{p,J}} = \inf_{\alpha\in\mathbb{R}} \frac{\left\|\int_{(c+d)/2}^{\cdot} u(t)dt - \alpha\right\|_{p,J}}{\|u\|_{p,J}} = \gamma_p |J|,$$

where

$$u(x) = \cos_p\left(\frac{\pi_p(x-(c+d)/2)}{d-c}\right) \text{ and } \gamma_p = (p')^{1/p} p^{1/p'} \pi^{-1} \sin(\pi/p)/2.$$

Proof. From Lemma 5.12 it follows that the function f of that lemma is odd with respect to $(c+d)/2$ and has a derivative vanishing at c and d; moreover, it is an extremal for

$$\sup_g \frac{\|\int_s^{\cdot} g(t)dt\|_{p,(s,d)}}{\|g\|_{p,(s,d)}} \text{ and } \sup_g \frac{\|\int_s^{\cdot} g(t)dt\|_{p,(c,s)}}{\|g\|_{p,(c,s)}}.$$

The result is now a consequence of Theorem 4.6. □

We also have from Theorem 4.6

Remark 5.2. The function ϕ defined by $\phi(x) = \sin_p\left(\pi_p \frac{x-a}{b-a}\right)$ satisfies

$$\frac{\|\phi\|_{p,I}}{\|\phi'\|_{p,I}} = \gamma_p |I|,$$

where γ_p is as in Theorem 5.8.

Three different partitions of $[a,b]$ will be useful in what follows. These are $J(n) := \{J_0, J_1, \dots, J_n\}$, where

$$J_0 = \left[a, a + \frac{b-a}{2n+1}\right], J_i = \left[a + \frac{(2i-1)(b-a)}{2n+1}, a + \frac{(2i+1)(b-a)}{2n+1}\right] \tag{5.4}$$

for $i = 1, \dots, n$; $S(n) := \{S_1, \dots, S_n\}$, where

$$S_i = \left[a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n}\right] \text{ for } i = 1, \dots, n, \tag{5.5}$$

and $I(n) := \{I_0, \dots, I_n\}$, where

$$\begin{gathered} I_0 = \left[a, a + \frac{b-a}{2n}\right], I_n = \left[b - \frac{b-a}{2n}, b\right], \\ I_i = \left[a + \frac{(b-a)}{2n}(2i-1), a + \frac{(b-a)}{2n}(2i+1)\right], \end{gathered} \tag{5.6}$$

for $i = 1, \dots, n-1$.

We first determine the approximation numbers (or linear widths) of the Hardy operator.

Lemma 5.13. *For all $n \in \mathbb{N}$,*

$$a_{n+1}(T_a) = \frac{\gamma_p |I|}{n + 1/2},$$

where γ_p is as in Theorem 5.8. Moreover, the bounded linear operators P_T, where

$$P_T f(x) = \sum_{i=1}^{n} \left(\int_a^{s_i} f(t) dt \right) \chi_{J_i}(x) + 0 \chi_{J_0}(x), \tag{5.7}$$

the J_i are given by (5.4) and s_i is the mid-point of J_i, are optimal n-dimensional linear approximations of T_a.

Proof. For simplicity we shall write T instead of T_a. Note that rank $(P_T) = n$. Put $J_i = [a_i, b_i]$; then $|[a_i, s_i]| = |[s_i, b_i]| = (b-a)/(2n+1)$ for $i = 1, \dots, n$. Let $f \in L_p(I)$. By Theorem 4.6,

$$\begin{gathered} \|Tf\|_{p,J_0} \le 2\left(\frac{b-a}{2n+1}\right) \gamma_p \|f\|_{p,J_0}, \\ \|Tf(\cdot) - Tf(s_i)\|_{p,(a_i,s_i)} \le 2\left(\frac{b-a}{2n+1}\right) \gamma_p \|f\|_{p,(a_i,s_i)} \end{gathered}$$

and

$$\|Tf(\cdot) - Tf(s_i)\|_{p,(s_i,b_i)} \le 2\left(\frac{b-a}{2n+1}\right)\gamma_p \|f\|_{p,(s_i,b_i)}$$

for $i = 1,\ldots,n$. Hence

$$\begin{aligned}
\|Tf - P_T f\|_{p,I}^p &= \sum_{i=1}^n \|f - (P_T f)(s_i)\|_{p,J_i}^p \\
&= \sum_{i=1}^n \left(\|Tf(\cdot) - Tf(s_i)\|_{p,(a_i,s_i)}^p + \|Tf(\cdot) - Tf(s_i)\|_{p,(s_i,b_i)}^p\right) \\
&\quad + \|Tf\|_{p,J_0}^p \\
&\le \left\{2\left(\frac{b-a}{2n+1}\right)\gamma_p\right\}^p \left\{\sum_{i=1}^n \left(\|f\|_{p,(a_i,s_i)}^p + \|f\|_{p,(s_i,b_i)}^p\right) + \|f\|_{p,J_0}^p\right\} \\
&\le \left\{2\left(\frac{b-a}{2n+1}\right)\gamma_p\right\}^p \|f\|_{p,I}^p .
\end{aligned}$$

Thus

$$a_{n+1}(T) \le \sup_{f \in L_p(I)} \left(\|Tf - P_T f\|_{p,I} / \|f\|_{p,I}\right) \le \frac{\gamma_p(b-a)}{n+1/2}.$$

To obtain an estimate for $a_{n+1}(T)$ from below we again use the partition $J(n)$ and take $\gamma \in (0,1)$. From Theorems 5.8 and 4.6 we see that there are functions ϕ_i, non-zero only on J_i, such that

$$\begin{aligned}
\inf_{c \in \mathbb{R}} \frac{\|T\phi_i - c\|_{p,J_i}}{\|\phi_i\|_{p,J_i}} &\ge \gamma\gamma_p |J_i| \text{ for } i = i,\ldots,n, \\
\frac{\|T\phi_0\|_{p,J_0}}{\|\phi_0\|_{p,J_0}} &\ge 2\gamma\gamma_p |J_0| .
\end{aligned}$$

Let $P : L_p(I) \to L_p(I)$ be bounded and linear, with rank n. Then there are constants $\lambda_0,\ldots,\lambda_n$, not all zero, such that for $g = \sum_{i=0}^n \lambda_i \phi_i$ we have $P_n g = 0$. Hence

$$\begin{aligned}
\|Tg - P_n g\|_{p,I}^p &= \|Tg\|_{p,I}^p \\
&= \sum_{i=0}^n \|Tg\|_{p,I_i}^p = \left\|\int_a^{\cdot} \lambda_0\phi_0\right\|_{p,J_0}^p + \sum_{i=1}^n \left\|\int_{a_i}^{\cdot} \lambda_i\phi_i + \int_a^{a_i} g\right\|_{p,J_i}^p \\
&\ge \left\|\int_a^{\cdot} \lambda_0\phi_0\right\|_{p,J_0}^p + \sum_{i=1}^n \inf_{\alpha \in \mathbb{R}} \left\|\int_{a_i}^{\cdot} \lambda_i\phi_i - \alpha\right\|_{p,J_i}^p \\
&\ge \left(\frac{\gamma\gamma_p |I|}{n+1/2}\right)^p |\lambda_0|^p \|\phi_0\|_{p,J_0}^p + \sum_{i=1}^n \left(\frac{\gamma\gamma_p |I|}{n+1/2}\right)^p |\lambda_i|^p \|\phi_i\|_{p,J_i}^p \\
&= \left(\frac{\gamma\gamma_p |I|}{n+1/2}\right)^p \|g\|_{p,I}^p .
\end{aligned}$$

Thus $a_{n+1}(T) \ge \gamma_p(b-a)/(n+1/2)$. The lemma follows. □

Theorem 5.9. *Let $T_a : L_p(I) \to L_p(I)$ be the Hardy operator given by (5.2) and let $\widetilde{s}_n$ stand for any strict s-number. Then for all $n \in \mathbb{N}$,*

$$\widetilde{s}_n(T_a) = \frac{\gamma_p\,|I|}{n-1/2}, \tag{5.8}$$

where γ_p is as in Theorem 5.8.

Proof. Let $n \in \mathbb{N}$. By Lemma 5.5, it is enough to show that $a_n(T_0) \le i_n(T_0)$ on $I = (0,1)$. We use the partition $I(n)$ given by (5.6). In this case this means that for each $i \in \{1,2,\dots,n\}$ we have $I_i = [a_{i-1}, a_i]$, where $a_0 = 0, \dots$ and $a_n = 1$. Note that when $i < n$, $|I_i| = 2\,|I_n|$. By $l^n_{p,w}$ we denote a sequence space with the norm

$$\|\{c_i\}_{i=i}^n\|_{l^n_{p,w}} := \left\{\sum_{i=1}^{n-1} 2\,|c_i|^p + |c_n|^p\right\}^{1/p}.$$

Maps $A : l^n_{p,w} \to L_p(0,1)$ and $B : L_p(0,1) \to l^n_{p,w}$ are defined by

$$A\left(\{c_i\}_{i=1}^n\right) = \sum_{i=1}^n (-1)^{i+1} c_i \chi_{I_i}(x) \cos_p\left((n-1/2)\pi_p x\right)$$

and

$$Bg(x) = \left\{\frac{(n-1/2)\pi_p \int_{I_i} (-1)^{i+1} g(x) (\sin_p\left[(n-1/2)\pi_p x\right])_{(p)} dx}{\left\|\sin_p\left((n-1/2)\pi_p x\right)\right\|_{p,I_i}^p}\right\}_{i=1}^n.$$

Since

$$T\left(c_i \chi_{I_i}(x) \cos_p\left((n-1/2)\pi_p x\right)\right) = \frac{c_i \sin_p\left((n-1/2)\pi_p x\right)}{(n-1/2)\pi_p} \chi_{I_i}(x),$$

we have

$$T\left(A\left(\{c_i\}_{i=1}^n\right)\right) = \sum_{i=1}^n \frac{(-1)^{i+1} c_i \sin_p\left((n-1/2)\pi_p x\right)}{(n-1/2)\pi_p} \chi_{I_i}(x).$$

Thus using the definition of B we obtain

$$B\left(T\left(A\left(\{c_i\}_{i=1}^n\right)\right)\right) = \left\{c_i \int_{I_i} \frac{\left|\sin_p\left((n-1/2)\pi_p t\right)\right|^p}{\left\|\sin_p\left((n-1/2)\pi_p x\right)\right\|_{p,I_i}^p}\right\}_{i=1}^n = \{c_i\}_{i=1}^n,$$

which means that BTA is the identity on $l^n_{p,w}$.

Moreover, $\left\{\left\|B : L_p(0,1) \to l^n_{p,w}\right\| / (n-1/2)\pi_p\right\}^p$ equals the supremum, over all $g \in L_p(0,1)$ with $\|g\|_{p,(0,1)} \leq 1$, of

$$2\sum_{i=1}^{n-1}\left|\frac{\int_{I_i}(-1)^{i+1}g(x)\left(\sin_p\left[(n-1/2)\pi_p x\right]\right)_{(p)}dx}{\left\|\sin_p\left((n-1/2)\pi_p x\right)\right\|^p_{p,I_i}}\right|^p + \left|\frac{\int_{I_n}(-1)^{n+1}g(x)\left(\sin_p\left[(n-1/2)\pi_p x\right]\right)_{(p)}dx}{\left\|\sin_p\left((n-1/2)\pi_p x\right)\right\|^p_{p,I_n}}\right|^p.$$

Note that the supremum is attained only when

$$g(x) = \sum_{i=1}^{n} c_i \chi_{I_i}(x)\sin_p\left((n-1/2)\pi_p x\right).$$

Hence $\left\|B : L_p(0,1) \to l^n_{p,w}\right\| / (n-1/2)\pi_p$ equals

$$\begin{aligned}
&\sup_{\{c_i\}\in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1}|c_i|^p + |c_n|^p\right)^{1/p}}{\left\|\sum_{i=1}^{n} c_i\chi_{I_i}(x)\sin_p\left((n-1/2)\pi_p x\right)\right\|_{p,(0,1)}} \\
&= \sup_{\{c_i\}\in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1}|c_i|^p + |c_n|^p\right)^{1/p}}{\left\{\sum_{i=1}^{n}\int_{I_i}\left|c_i\chi_{I_i}(x)\sin_p\left((n-1/2)\pi_p x\right)\right|^p dx\right\}^{1/p}} \\
&= \sup_{\{c_i\}\in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1}|c_i|^p + |c_n|^p\right)^{1/p}}{\left(2\sum_{i=1}^{n-1}|c_i|^p + |c_n|^p\right)^{1/p}\left\{\int_{I_n}\left|\sin_p\left((n-1/2)\pi_p x\right)\right|^p dx\right\}^{1/p}} \\
&= \frac{1}{\left\{\int_{I_n}\left|\sin_p\left((n-1/2)\pi_p x\right)\right|^p dx\right\}^{1/p}},
\end{aligned}$$

and $\left\|A : l^n_{p,w} \to L_p(0,1)\right\|$ equals

$$\begin{aligned}
&\sup_{\|\{c_i\}\|_{l^n_{p,w}}\leq 1}\left\{\int_I\left|\sum_{i=1}^{n} c_i\chi_{I_i}(x)\cos_p\left((n-1/2)\pi_p x\right)\right|^p dx\right\}^{1/p} \\
&= \sup_{\|\{c_i\}\|_{l^n_{p,w}}\leq 1}\left\{\sum_{i=1}^{n}|c_i|^p\int_{I_i}\left|\cos_p\left((n-1/2)\pi_p x\right)\right|^p dx\right\}^{1/p} \\
&= \sup_{\|\{c_i\}\|_{l^n_{p,w}}\leq 1}\left(2\sum_{i=1}^{n-1}|c_i|^p + |c_n|^p\right)^{1/p}\left(\int_{I_n}\left|\cos_p\left((n-1/2)\pi_p x\right)\right|^p dx\right)^{1/p} \\
&= \left(\int_{I_n}\left|\cos_p\left((n-1/2)\pi_p x\right)\right|^p dx\right)^{1/p}.
\end{aligned}$$

Thus

$$i_n(T) \geq \|A\|^{-1}\|B\|^{-1} = \frac{\left(\int_{I_n} \left|\sin_p\left((n-1/2)\pi_p x\right)\right|^p dx\right)^{1/p}}{(n-1/2)\pi_p \left(\int_{I_n} \left|\cos_p\left((n-1/2)\pi_p x\right)\right|^p dx\right)^{1/p}},$$

which completes the proof. □

Next we focus on the Hardy operator T_c when $c = (a+b)/2$.

Lemma 5.14. *Let n be an odd natural number and let $c = (a+b)/2$. Then*

$$a_{n+1}(T_c) = a_n(T_c) = \gamma_p |I|/n,$$

where γ_p is as in Theorem 5.8. Moreover, the bounded linear operator P_{T_c} defined by

$$P_{T_c} f(x) = \sum \left(\int_c^{d_i} f(t)dt \right) \chi_{S_i}(x) + 0\chi_{S_{(n+1)/2}}(x), \tag{5.9}$$

where the sum is over all $i \in \{1,2,\ldots,n\}$ with $i \neq (n+1)/2$, $S(n) = \{S_i\}_{i=1}^n$ is the partition of $[a,b]$ given by (5.5) and d_i is the mid-point of S_i, is the optimal linear approximant to T_c among all n- and $(n-1)$-dimensional linear operators.

Proof. Let $S_i = [a_i, b_i]$, so that $d_i = (a_i+b_i)/2$, and note that $|S_i| = |I|/n$. The map P_{T_c} given by (5.9) has rank $n-1$. Let $f \in L_p(I)$. By Theorem 4.6, $\left(\frac{b-a}{n}\right)\gamma_p \|f\|_{p,(d_i,b_i)}$ is greater than or equal to

$$\max\left\{ \left\| \int_{d_i}^{\cdot} f(t)dt \right\|_{p,(d_i,b_i)}, \left\| \int_{d_i}^{\cdot} f(t)dt \right\|_{p,(a_i,d_i)} \right\} \text{ if } i \neq (n+1)/2,$$

and

$$\max\left\{ \|T_c f\|_{p,(d_i,b_i)}, \|T_c f\|_{p,(a_i,d_i)} \right\} \text{ if } i = (n+1)/2.$$

From this we obtain, as in the previous lemma,

$$\|T_c f - P_{T_c} f\|_{p,I}^p \leq \left\{\gamma_p (b-a)/n\right\}^p \|f\|_{p,I}^p,$$

so that for odd n we have

$$a_n(T_c) \leq \gamma_p |I|/n.$$

To estimate the approximation numbers from below we again use the partition $S(n) = \{S_i\}_{i=1}^n$ of I, with $S_i = [a_i, b_i]$, $b_i - a_i = |I|/n$; the mid-point of S_i is d_i and $d_{(n+1)/2} = c$. Let $\gamma \in (0,1)$. Then using Theorems 5.8 and 4.6 for each $i \in \{1,2,\ldots,n\}, i \neq (n+1)/2$, we see that there are functions $\phi \in L_p(I)$, non-zero only on S_i, and functions $\phi_-, \phi_+ \in L_p(I)$, non-zero only on $(a_{(n+1)/2}, c)$ and $(c, b_{(n+1)/2})$ respectively, such that

$$\inf_{\alpha\in\mathbb{R}} \frac{\|T_c\phi_i - \alpha\|_{p,S_i}}{\|\phi_i\|_{p,S_i}}, \frac{\|T_c\phi_-\|_{p,(a_{(n+1)/2},c)}}{\|\phi_-\|_{p,(a_{(n+1)/2},c)}} \text{ and } \frac{\|T_c\phi_+\|_{p,(c,b_{(n+1)/2})}}{\|\phi_+\|_{p,(c,b_{(n+1)/2})}}$$

are all greater than or equal to $\gamma\gamma_p |S_i|$. Let $P_n : L_p(I) \to L_p(I)$ be bounded and linear, with rank n. Then there are constants λ_i $(i \in \{1,2,\dots,n\}, i \neq (n+1)/2)$, λ_-, λ_+ such that for $g = \sum \lambda_i\phi_i + \lambda_-\phi_- + \lambda_+\phi_+$ we have $P_n g = 0$. As in the previous lemma we obtain

$$\|T_c g - P_n g\|_{p,I}^p \geq (\gamma\gamma_p |I|/n)^p \|g\|_{p,I}^p,$$

from which it follows that for odd n, $a_{n+1}(T_c) \geq \gamma_p |I|/n$. Hence for odd n,

$$\gamma_p |I|/n \leq a_{n+1}(T_c) \leq a_n(T_c) \leq \gamma_p |I|/n,$$

and the proof is complete. □

Lemma 5.15. *Let n be an odd natural number and let $T_c : L_p(I) \to L_p(I)$ be the Hardy operator with $c = (a+b)/2$. Then*

$$\gamma_p \frac{|I|}{n} = a_{n+1}(T_c) \leq i_{n+1}(T_c),$$

where γ_p is as in Theorem 5.8.

Proof. It is enough to deal with the case when $I = (-1,1)$. For $i = -(n-1)/2, \dots, (n-1)/2$ define $a_i = 2i/n$; set $a_{(n+1)/2} = 1$ and $a_{-(n+1)/2} = -1$. Introduce a covering of $(-1,1)$ by means of the $(n+1)$ non-overlapping intervals $I_i := (a_{i-1}, a_i)$, where $i = -(n-1)/2, \dots, (n+1)/2$. Note that $2\left|I_{(n+1)/2}\right| = 2\left|I_{-(n-1)/2}\right| = |I_i|$ when $-(n-1)/2 + 1 < i < (n+1)/2$.

As in Theorem 5.9 we introduce a sequence space $l_{p,w}^{(n+1)/2}$ with norm

$$\|\{c_i\}\|_{l_{p,w}^{(n+1)/2}} := \left\{ 2 \sum_{i=-(n-1)/2+1}^{(n+1)/2-1} |c_i|^p + \left|c_{(n+1)/2}\right|^p + \left|c_{-(n-1)/2}\right|^p \right\}^{1/p}.$$

Maps $A : l_{p,w}^{(n+1)/2} \to L_p(0,1)$ and $B : L_p(0,1) \to l_{p,w}^{(n+1)/2}$ are defined by

$$A\left(\{c_i\}_{i=-(n-1)/2}^{(n+1)/2}\right) = \sum_{i=-(n-1)/2}^{(n+1)/2} (-1)^{i+1} c_i \chi_{I_i}(x) \cos_p(\pi_p(n-1/2)x)$$

and

$$B(g(x)) = \left\{ \frac{(n-1/2)\pi_p \int_{I_i} (-1)^{i+1} g(x) \left(\sin_p[\pi_p(n-1/2)x]\right)_{(p)} dx}{\left\|\sin_p(\pi_p(n-1/2)x)\right\|_{p,I_i}} \right\}_{i=-(n-1)/2}^{(n+1)/2}.$$

Then

$$T\left(c_i \chi_{I_i}(x) \cos_p(\pi_p(n-1/2)x)\right) = \frac{c_i \chi_{I_i}(x) \sin_p(\pi_p(n-1/2)x)}{\pi_p(n-1/2)},$$

from which it follows that

$$T\left(A\left(\{c_i\}_{i=-(n-1)/2}^{(n+1)/2}\right)\right) = \sum_{i=-(n-1)/2}^{(n+1)/2} \frac{(-1)^{i+1} c_i \chi_{I_i}(x) \sin_p(\pi_p(n-1/2)x)}{\pi_p(n-1/2)}.$$

Using the definition of B we obtain

$$\begin{aligned} B\left(T\left(A\left(\{c_i\}_{i=-(n-1)/2}^{(n+1)/2}\right)\right)\right) &= \left\{ c_i \int_{I_i} \frac{\left|\sin_p(\pi_p(n-1/2)t)\right|^p}{\left\|\sin_p(\pi_p(n-1/2)x)\right\|_{p,I_i}} dt \right\}_{i=-(n-1)/2}^{(n+1)/2} \\ &= \{c_i\}_{i=-(n-1)/2}^{(n+1)/2}. \end{aligned}$$

Thus BTA is the identity on $l_{p,w}^{(n+1)/2}$. The rest of the proof is a simple modification of that of Theorem 5.9. □

From Lemmas 5.15 and 5.14 we have

Theorem 5.10. *Let $T_c : L_p(I) \to L_p(I)$ be the Hardy operator with $c = (a+b)/2$ and let $\widetilde{s}_n$ stand for any strict s-number. If n is odd, then*

$$\widetilde{s}_{n+1}(T_c) = \widetilde{s}_n(T_c) = \gamma_p |I| / n, \tag{5.10}$$

where γ_p is as in Theorem 5.8. The bounded linear operator P_{T_c} defined in (5.9) is an optimal n-dimensional approximation of T_c.

5.3 s-Numbers of the Sobolev Embedding on Intervals

Here we study the behaviour of s-numbers of various Sobolev embeddings. Generalized trigonometric functions will play an essential role in obtaining the exact values of different s-numbers.

Through this section $I = [a,b]$ will be an interval with $-\infty < a < b < \infty$ and $\mathbb{T}$ will be the unit circle realized as an interval $[-\pi, \pi]$ with identified points $-\pi$ and π.

By $W_p^1(\mathbb{T})$ (or respectively by $W_p^1(I)$) we understand the Sobolev space of functions on $\mathbb{T}$ (or on I) (i.e. the set of all absolutely continuous functions on $\mathbb{T}$ with $\|f'\|_{p,\mathbb{T}} < \infty$, or respectively on I with $\|f'\|_{p,I} < \infty$). Note that $\|f'\|_{p,\mathbb{T}}$ and $\|f'\|_{p,I}$ are pseudonorms on $W_p^1(\mathbb{T})$ or $W_p^1(I)$.

As usual, $\overset{\circ}{W_p^1}(I)$ is the space of all absolutely continuous functions on I with finite norm $\|f'\|_{p,I}$ and 0 boundary values at a and b.

By $\overset{a}{W_p^1}(I)$ (or respectively by $\overset{mid}{W_p^1}(I)$) we mean the space of all absolutely continuous functions on I with finite norm $\|f'\|_{p,I}$ and 0 boundary value at a (or with 0 value at the middle of the interval I).

For $1 < p < \infty$ we shall consider in this section the following Sobolev embeddings

$$E_0 : \overset{\circ}{W_p^1}(I) \to L_p(I),$$
$$E_a : \overset{a}{W_p^1}(I) \to L_p(I),$$
$$E_{mid} : \overset{mid}{W_p^1}(I) \to L_p(I),$$

and their variants:

$$E_1 : W_p^1(I)/\operatorname{sp}\{1\} \to L_p(I)/\operatorname{sp}\{1\},$$
$$E_2 : W_p^1(\mathbb{T})/\operatorname{sp}\{1\} \to L_p(\mathbb{T})/\operatorname{sp}\{1\}.$$

By $W_p^1(I)/\operatorname{sp}\{1\}$ we denote the factorization of the space $W_p^1(I)$ with respect to constants, equipped with the norm $\|f'\|_p$. Note $f \in W_p^1(I)/\operatorname{sp}\{1\}$ if and only if $\|f\|_{p,I} = \inf_{c\in\mathbb{R}} \|f-c\|_{p,I}$ and $\|f'\|_{p,I} < \infty$. In a similar way we define $L_p(I)/\operatorname{sp}\{1\}$, $W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$ and $L_p(\mathbb{T})/\operatorname{sp}\{1\}$.

The norms of the embedding E_0 is defined by

$$\|E_0\| = \sup_{\|f'\|_{p,I}>0, f(a)=f(b)=0} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}};$$

those of E_a and E_{mid} we define in a similar way, while that of E_1 is given by:

$$\|E_1\| = \sup_{f\in W_p^1(I)/\operatorname{sp}\{1\}} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}},$$

and likewise we define the norm of E_2.

Since $|I| < \infty$ it is well-known that, all these embeddings are compact (see for example, [41], Theorem V.4.18).

By $B\overset{a}{W_p^1}(I) = \{f; f \in \overset{a}{W_p^1}(I) \text{ and } \|f'\|_{p,I} \le 1\}$, and $BL^p(I) = \{f; f \in L^p(I) \text{ and } \|f'\|_{p,I} \le 1\}$ we denote the unit balls in $\overset{\circ}{W_p^1}(I)$, and $L^p(I)$ respectively; unit balls in other spaces are denoted by similar expressions. Obviously we have

$$T_a(BL^p(I)) = B\overset{a}{W^1_p}(I)$$

$$T_c(BL^p(I)) = B\overset{mid}{W^1_p}(I)$$

where $c = (a+b)/2$. From this observation and Theorems 5.9 and 5.10 the next theorem follows.

Theorem 5.11. *Let $n \in \mathbb{N}$ and let $\widetilde{s}_n$ stand for any strict s-number.*

(i) *If n is odd, then $\widetilde{s}_n(E_{mid}) = \widetilde{s}_{n+1}(E_{mid}) = \gamma_p \frac{|I|}{n}$.*
(ii) *For all $n \in N$, then $\widetilde{s}_n(E_a) = \gamma_p \frac{|I|}{n+1/2}$,*

where γ_p is as in Theorem 5.8.

Next we shall focus our interest on the strict s-numbers for the Sobolev embeddings on $\mathbb{T}$ and on I. At first we consider the Sobolev embedding E_2.

Theorem 5.12. *Let $n \in \mathbb{N}$ and let $\widetilde{s}_n$ stand for any strict s-number. If n is even, then*

$$\widetilde{s}_n(E_2) \geq \gamma_p \frac{2\pi}{n+1},$$

and when n is odd,

$$\widetilde{s}_n(E_2) = \gamma_p \frac{2\pi}{n+1},$$

where γ_p is as in Theorem 5.8. Moreover, for given odd n, the bounded linear operator $P_{\mathbb{T}}$ given by

$$P_{\mathbb{T}} f(x) = \sum_{i=1}^{n+1} \frac{f(a_i)+f(b_i)}{2} \chi_{S_i}(x) \tag{5.11}$$

where $\{S_i\}_1^{n+1} = S(n+1)$ is a partition of $I = [a,b] = \mathbb{T} = [-\pi,\pi]$ (see (5.5) with $S_i = [a_i,b_i]$, $a_0 = b_n$, and $a_{i+1} = b_i$), is an optimal linear operator for the Sobolev embedding E_2 among all linear operators with rank $\leq n-1$.

Proof. Let n be odd and $\{S_i\}_{i=1}^{n+1} = S(n+1)$ be a partition of $[-\pi,\pi] = \mathbb{T} = I = [a,b]$. We can rewrite the operator $P_{\mathbb{T}}$ in the following way:

$$\begin{aligned} P_{\mathbb{T}} f(x) = {} & \frac{f(a_1)+f(b_1)}{2} \chi_{\mathbb{T}}(x) \\ & + \sum_{i=2}^{n} \left(\frac{[f(a_i)+f(b_i)]}{2} - \frac{[f(a_1)+f(b_1)]}{2} \right) \chi_{S_i}(x) \\ & + \left(\left[\sum_{i=1}^{n} [f(a_i)+f(b_i)](-1)^i \frac{1}{2} \right] - \frac{f(a_1)+f(b_1)}{2} \right) \chi_{S_{n+1}}(x). \end{aligned}$$

From this we can see that the rank of $P_{\mathbb{T}}$ as a linear operator from $W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$ into $L^p(\mathbb{T})/\operatorname{sp}\{1\}$ is equal to $n-1$. Let $f \in W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$; then

$$\inf_{c\in\mathbb{R}} \|f - P_{\mathbb{T}}f - c\|_{p,\mathbb{T}}^p \le \|f - P_{\mathbb{T}}f\|_{p,\mathbb{T}}^p = \sum_{i=1}^{n+1} \|f - \frac{f(a_i)+f(b_i)}{2}\|_{p,S_i}^p.$$

From Lemma 5.11 we have for any i with $1 \le i \le n+1$:

$$\begin{aligned}
\sup_{\|f\|_{W_p^1(S_i)}\le 1} \|f - \frac{f(a_i)+f(b_i)}{2}\|_{p,S_i}^p &= \sup_{\|f\|_{W_p^1(S_i)\le 1}} \inf_{c\in\mathbb{R}} \|f - c\|_{p,S_i}^p \\
&= \sup_{\|f\|_{W_p^1(S_i)}\le 1} \inf_{c\in\mathbb{R}} \|f - \frac{f(a_i)+f(b_i)}{2} - c\|_{p,S_i}^p \\
&= \sup_{\|f\|_{W_p^1(S_i)}\le 1} (\gamma_p |S_i|)^p \, \|f'\|_{p,S_i}^p,
\end{aligned}$$

and then

$$\|f - P_{\mathbb{T}}f\|_{L^p(\mathbb{T})/\operatorname{sp}\{1\}} \le \gamma_p \left(\frac{2\pi}{n+1}\right) \|f'\|_{p,\mathbb{T}}.$$

Thus

$$a_n(E_2) \le \gamma_p \frac{2\pi}{n+1}.$$

To prove the lower estimate for $i_n(E_2)$, we introduce a sequence space $l_p^{n+1}/\operatorname{sp}\{1\}$ with norm

$$\|\{c_i\}\|_{l_p^{n+1}/\operatorname{sp}\{1\}} := \inf_{c\in\mathbb{R}} \left\{\sum_{i=1}^{n+1} |c_i - c|^p\right\}^{1/p}.$$

Note that $\dim l_p^{n+1}/\operatorname{sp}\{1\} = n$.

Define a map $A : l_p^{n+1}/\operatorname{sp}\{1\} \to W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$ by:

$$A\left(\{c_i\}_{i=1}^{n+1}\right) = \sum_{i=1}^{n+1} (c_i - c)\chi_{I_i}(x) \sin_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right) + c,$$

where c is a number for which

$$\|\{c_i\}\|_{l_p^{n+1}/\operatorname{sp}\{1\}} = \left\{\sum_{i=1}^{n+1} |c_i - c|^p\right\}^{1/p}.$$

Similarly, a map $B : L_p(\mathbb{T})/\operatorname{sp}\{1\} \to l_p^{n+1}/\operatorname{sp}\{1\}$ is defined by

$$Bg(x) = \left\{ \frac{\int_{I_i} (g(x) - c)\, (\sin_p [(x - a_i)(n+1)\pi_p/(2\pi)])_{(p)}}{\left\| \sin_p ((x - a_i)(n+1)\pi_p/(2\pi)) \right\|_{p,I_i}^p} + c \right\}_{i=1}^{n+1},$$

where c is a constant such that $\|g\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}} = \|g - c\|_{p,\mathbb{T}}$.

Since $E_2(g(x)) = g(x)$ we have $E_2(A(\{c_i\}_{i=1}^{n+1})) = A(\{c_i\}_{i=1}^{n+1})$.

Thus using the definition of B we obtain

$$\begin{aligned} B\left(E_2\left(A\left(\{c_i\}_{i=1}^{n+1}\right)\right)\right) &= \left\{ c_i \int_{I_i} \frac{\left|\sin_p [(x - a_i)(n+1)\pi_p/(2\pi)]\right|^p}{\left\| \sin_p [(x - a_i)(n+1)\pi_p/(2\pi)] \right\|_{p,I_i}^p} \right\}_{i=1}^{n+1} \\ &= \{c_i\}_{i=1}^{n+1}, \end{aligned}$$

which means that BE_2A is the identity on $l_p^{n+1}/\operatorname{sp}\{1\}$.

Moreover, $\left\|B : L_p(\mathbb{T})/\operatorname{sp}\{1\} \to l_p^{n+1}/\operatorname{sp}\{1\}\right\|^p$ equals the supremum, over all $g \in L_p(\mathbb{T})/\operatorname{sp}\{1\}$ with $\|g\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}} \leq 1$, of

$$\sum_{i=1}^{n+1} \left| \frac{\int_{I_i} (g(x) - c)\, (\sin_p ((n+1)\pi_p x/(2\pi)))_{(p)}\, dx}{\left\| \sin_p ((n+1)\pi_p x/(2\pi)) \right\|_{p,I_i}^p} \right|^p,$$

where c depends on g in such a way that $\|g\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}} = \|g - c\|_{p,\mathbb{T}}$. Note that then the supremum is attained only when

$$g(x) - c = \sum_{i=1}^{n+1} c_i \chi_{I_i}(x) \sin_p ((n+1)\pi_p x/(2\pi))$$

where c depends on g as above and $\|\{c_i\}_{i=1}^{n+1}\|_{l_p^{n+1}/\operatorname{sp}\{1\}} = \|\{c_i\}_{i=1}^{n+1}\|_{l_p^{n+1}}$. Then

$$\begin{aligned} &\left\|B : L_p(\mathbb{T})/\operatorname{sp}\{1\} \to l_p^{n+1}/\operatorname{sp}\{1\}\right\| \\ &\leq \sup_{\{c_i\} \in l_p^{n+1}} \frac{\left(\sum_{i=1}^{n+1} |c_i|^p\right)^{1/p}}{\left\| \sum_{i=1}^{n+1} c_i \chi_{I_i}(x) \sin_p ((n+1)\pi_p x/(2\pi)) \right\|_{p,\mathbb{T}}} \\ &= \sup_{\{c_i\} \in l_p^{n+1}} \frac{\left(\sum_{i=1}^{n+1} |c_i|^p\right)^{1/p}}{\left\{ \sum_{i=1}^{n+1} \int_{I_i} \left| c_i \chi_{I_i}(x) \sin_p ((n+1)\pi_p x/(2\pi)) \right|^p dx \right\}^{1/p}} \\ &= \sup_{\{c_i\} \in l_p^{n+1}} \frac{\left(\sum_{i=1}^{n+1} |c_i|^p\right)^{1/p}}{\left(\sum_{i=1}^{n+1} |c_i|^p\right)^{1/p} \left\{ \int_{I_1} \left| \sin_p ((n+1)\pi_p x/(2\pi)) \right|^p dx \right\}^{1/p}} \\ &= \left\{ \int_{I_1} \left| \sin_p ((n+1)\pi_p x/(2\pi)) \right|^p dx \right\}^{-1/p}, \end{aligned}$$

and $\left\|A : l_p^{n+1}/\operatorname{sp}\{1\} \to W_p^1(\mathbb{T})/\operatorname{sp}\{1\}\right\|$ equals

$$\sup_{\|\{c_i\}\|_{l_p^{n+1}/\operatorname{sp}\{1\}}\le 1} \left\{ \int_I \sum_{i=1}^{n+1} \left| (c_i - c)\chi_{I_i}(x) \frac{d}{dx}\left[\sin_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right]\right|^p dx \right\}^{1/p}$$

$$= \sup_{\|\{c_i\}\|_{l_p^{n+1}/\operatorname{sp}\{1\}}\le 1} \left\{ \sum_{i=1}^{n+1} |c_i - c|^p \int_{I_i} \left|\cos_p\left(\frac{(x-a_i)(n+1)\pi_p}{2\pi}\right)\left(\frac{(n+1)\pi_p}{2\pi}\right)\right|^p dx \right\}^{1/p}$$

$$= \left(\frac{(n+1)\pi_p}{2\pi}\right)\left\{\int_{I_1}\left|\cos_p\left(\frac{(x-a_1)(n+1)\pi_p}{2\pi}\right)\right|^p dx\right\}^{1/p}.$$

Thus

$$i_n(E_2) \ge \|A\|^{-1}\|B\|^{-1} = \frac{2\pi\left(\int_{I_1}\left|\sin_p((n+1)\pi_p(x-a_1)/(2\pi))\right|^p dx\right)^{1/p}}{(n+1)\pi_p\left(\int_{I_1}\left|\cos_p((n+1)\pi_p(x-a_1)/(2\pi))\right|^p dx\right)^{1/p}},$$

which completes the proof.

When n is even, by using the above techniques we obtain

$$i_n(E_2) \ge \gamma_p \frac{2\pi}{n+1}.$$

□

Now we focus on the Sobolev embedding E_1 on an interval I.

Theorem 5.13. *Let $n \in \mathbb{N}$ and let $\widetilde{s}_n$ stand for any strict s-number. Then*

$$\widetilde{s}_n(E_1) = \gamma_p \frac{|I|}{n},$$

where γ_p is as in Theorem 5.8.

Proof. This is obtained by using ideas from the proof of Theorem 5.12. □

Theorem 5.14. *Let $n \in \mathbb{N}$ and $\widetilde{s}_n$ stand for any strict s-number. Then*

$$\widetilde{s}_n(E_0) = \gamma_p \frac{|I|}{n},$$

where γ_p is as in Theorem 5.8.

Proof. Let $I(n) = \{I_i\}_{i=0}^n$ be a partition of $I = [a,b]$ (see (5.6)) with $I_i = [a_i, b_i]$, $a_0 = a$, $b_n = b$ and $a_{i+1} = b_i$. Clearly $2|I_0| = 2|I_n| = |I_i| = |I|/n$ for $i = 1, \dots, n-1$. We define an operator P_{n-1} with $\operatorname{rank}(P_{n-1}) = n-1$ by:

$$P_{n-1}f(x) := 0\chi_{I_0}(x) + 0\chi_{I_n} + \sum_{i=1}^{n-1} f\left(\frac{a_i+b_i}{2}\right)\chi_{I_i}(x).$$

Thus using Theorem 5.8 we have

$$\begin{aligned}
(a_n(E_0))^p &\le \sup_{f\in \overset{\circ}{W}{}^1_p(I)} \|(E_0-P_{n-1})(f)\|^p_{L^p(I)} \\
&\le \sup_{f\in \overset{\circ}{W}{}^1_p(I)} \left(\left[\sum_{i=1}^{n-1} \|f(.)-f\left(\frac{a_i+b_i}{2}\right)\|^p_{p,I_i} \right] + \|f\|^p_{p,I_0} + \|f\|^p_{p,I_n} \right) \\
&\le \sup_{\|u\|_{p,I}\le 1} \left(\left[\sum_{i=1}^{n-1} \| \int_{\frac{a_i+b_i}{2}}^{.} u(t)dt\|^p_{p,I_i} \right] + \| \int_a^{.} u(t)dt\|^p_{p,I_0} + \| \int_{.}^{b} u(t)dt\|^p_{p,I_n} \right) \\
&\le \sup_{\|u\|_{p,I}\le 1} \left(\left[\sum_{i=1}^{n-1} (\gamma_p|I_i|)^p\|u\|^p_{p,I_i} \right] + (\gamma_p 2|I_0|)^p\|u\|^p_{p,I_0} + (\gamma_p 2|I_n|)^p\|u\|^p_{p,I_n} \right) \\
&\le \left[\gamma_p \frac{|I|}{n}\right]^p,
\end{aligned}$$

and then $a_n(E_0) \le \gamma_p|I|/n$.

Now we shall prove the lower estimate for $i_n(E_0)$. The map $A : l^n_p \to \overset{\circ}{W}{}^1_p(I)$ is defined by:

$$A\left(\{c_i\}_{i=1}^n\right) = \sum_{i=1}^n c_i \chi_{S_i}(x) \sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right),$$

where $\{S_i\}_{i=1}^n$ is a partition of I (see (5.5)) with $S_i = [a_i,b_i]$ and $|S_i| = |I|/n$. The map $B : L_p(I) \to l^n_p$ is defined by

$$Bg(x) = \left\{ \frac{\int_{S_i} g(x)\left(\sin_p\left[(x-a_i)n\pi_p/|I|\right]\right)_{(p)}}{\left\|\sin_p\left((x-a_i)n\pi_p/|I|\right)\right\|^p_{p,S_i}} \right\}_{i=1}^n.$$

Obviously we have $E_0(A(\{c_i\}_{i=1}^n)) = A(\{c_i\}_{i=1}^n)$ and then

$$B\left(E_0\left(A\left(\{c_i\}_{i=1}^n\right)\right)\right) = \left\{ c_i \int_{S_i} \frac{\left|\sin_p\left[(x-a_i)n\pi_p/|I|\right]\right|^p}{\left\|\sin_p\left[(x-a_i)n\pi_p/|I|\right]\right\|^p_{p,S_i}} \right\}_{i=1}^n = \{c_i\}_{i=1}^n,$$

which means that BE_0A is the identity on l^n_p.

Note that $\left\|B : L_p(I) \to l^n_p\right\|$ equals the supremum of $\|Bg|l^n_p\|$, over all $g \in L_p(I)$ with $\|g\|_{L_p(I)} \le 1$, and the supremum is attained only when

$$g(x) = \sum_{i=1}^n c_i \chi_{S_i}(x) \sin_p\left(n\pi_p x/|I|\right).$$

Then we have

$$\left\|B : L_p(I) \to l_p^n\right\| \le \sup_{\{c_i\}\in l_p^n} \frac{\left(\sum_{i=1}^n |c_i|^p\right)^{1/p}}{\left\|\sum_{i=1}^n c_i \chi_{S_i}(x) \sin_p(n\pi_p x/|I|)\right\|_{p,I}}$$
$$= \left\{\int_{S_1} \left|\sin_p(n\pi_p x/|I|)\right|^p dx\right\}^{-1/p},$$

and $\left\|A : l_p^n \to \overset{\circ}{W}{}_p^1(I)\right\|$ equals

$$\sup_{\|\{c_i\}\|_{l_p^n}\le 1} \left\{\int_I \sum_{i=1}^n \left| c_i \chi_{S_i}(x) \frac{d}{dx}\left[\sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\right]\right|^p dx\right\}^{1/p}$$
$$= \sup_{\|\{c_i\}\|_{l_p^n}\le 1} \left\{\sum_{i=1}^n |c_i|^p \int_{S_i} \left|\cos_p\left(\frac{(x-a_i)n\pi_p}{|I|}\right)\left(\frac{n\pi_p}{|I|}\right)\right|^p dx\right\}^{1/p}$$
$$= \left(\frac{n\pi_p}{|I|}\right)\left\{\int_{S_1} \left|\cos_p\left(\frac{(x-a_1)n\pi_p}{|I|}\right)\right|^p dx\right\}^{1/p}.$$

Thus

$$i_n(E_0) \ge \|A\|^{-1}\|B\|^{-1} = \frac{|I|\left(\int_{S_1}\left|\sin_p(n\pi_p(x-a_1)/|I|)\right|^p dx\right)^{1/p}}{n\pi_p\left(\int_{S_1}\left|\cos_p((n+1)\pi_p(x-a_1)/(2\pi))\right|^p dx\right)^{1/p}},$$

which completes the proof. □

Note that we can use generalised trigonometric functions to obtain some insight into the shape of the unit ball of Sobolev spaces. In particular, Theorem 5.14 provides us with information about the image of the unit ball of $\overset{\circ}{W}{}_p^1(I)$ in the space $L_p(I)$. It implies that the largest element in $B\overset{\circ}{W}{}_p^1(I) := \{f; \|f|\overset{\circ}{W}{}_p^1\| \le 1\}$ in the $L_p(I)$ norm is

$$f_1(x) := \frac{\sin_p(\pi_p(x-a)/|I|)}{\|\sin_p(\pi_p(x-a)/|I|)\,|\,\overset{\circ}{W}{}_p^1(I)\|}.$$

When we approximate $B\overset{\circ}{W}{}_p^1(I)$ by a one-dimensional subspace in $L_p(I)$, the most distant element from the optimal one-dimensional approximation is

$$f_2(x) := \frac{\sin_p(2\pi_p(x-a)/|I|)}{\|\sin_p(2\pi_p(x-a)/|I|)\,|\,\overset{\circ}{W}{}_p^1(I)\|}.$$

More generally, if we approximate $B\overset{\circ}{W}{}_p^1(I)$ by an n-dimensional subspace in $L_p(I)$, then the most distant element from the optimal n-dimensional approximation is

$$f_n(x) := \frac{\sin_p(n\pi_p(x-a)/|I|)}{\|\sin_p(n\pi_p(x-a)/|I|)\mid \overset{\circ}{W}{}_p^1(I)\|}.$$

Also from the previous theorem we have that $\|f_i\|_{L_p(I)} = s_n(E_0)$

We can see that the functions f_i play, in some sense, rôles similar to those of the semi-axes of an ellipsoid.

We present below figures which show an image of $B\overset{\circ}{W}{}_p^1(I)$ restricted to the linear subspace span$\{f_1, f_2, f_3\}$ in $L_p(I)$.

In the case $p = 2$ we obtain an ellipsoid (here the x, y, z axes correspond to f_1, f_2, f_3).

When $p = 10$ and $p = 1.1$ we have the images below:

We can see that the main difference between Figs. 5.1 and 5.2 is that the pictures in Fig. 5.2 are not convex. This suggests that possibly the functions f_1, f_2, f_3 are not orthogonal in the James sense.

Notes

Note 5.1. Most of the material in this section is quite standard. The books by Pietsch [106, 107] and Pinkus [109] should be consulted for further details and background information.

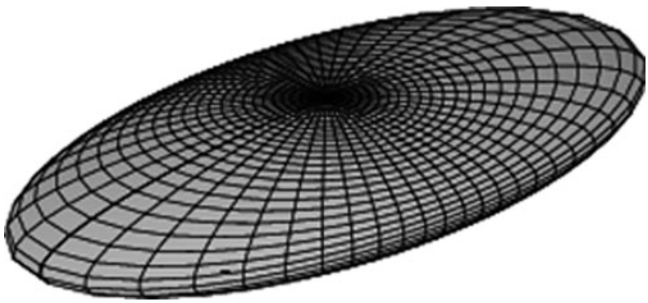

Fig. 5.1 $p = 2$

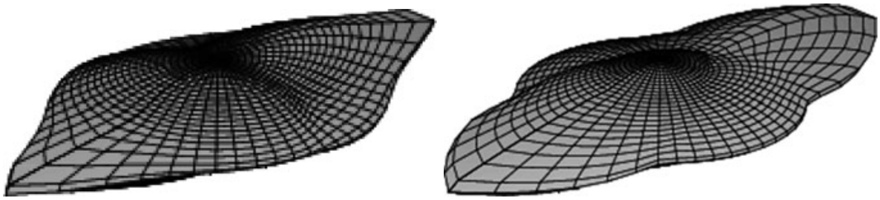

Fig. 5.2 $p = 10$ $p = 1.1$

Note 5.2. The very precise information about the strict s-numbers of the various Hardy operators contained in Theorems 5.5 and 5.1 was first obtained in [46, 47].

Note 5.3. The exact determination of the approximation numbers of the Sobolev embeddings was carried out in [47] and also in [60] and [87]. For the results concerning the strict s-numbers see [47].

Chapter 6
Estimates of *s*-Numbers of Weighted Hardy Operators

It is shown that if $1 < p < \infty$ and the Hardy operator T is viewed as a map from $L_p(a,b)$ to itself, then all strict s-numbers of T coincide and their asymptotic behaviour is determined. The cases $p = 1$ or ∞ require separate treatment and less is proved, but upper and lower estimates of the approximation numbers of T are obtained.

6.1 Introduction and Basic Notation

In this chapter we consider the map $T_{a,(a,b),v,u}$ of Hardy type defined by:

$$T_{a,(a,b),v,u}f(x) = v(x)\int_a^x u(t)f(t)dt, \tag{6.1}$$

where u and v are given real-valued functions with $|\{x : u(x) = 0\}| = |\{x : v(x) = 0\}| = 0$. This map will act between Lebesgue spaces on an interval (a,b). If no ambiguity is likely we shall simply denote this map by $T_{a,(a,b)}$ or T.

Throughout this section we shall assume that $-\infty < a < b < \infty$ and $1 \le p \le \infty$, and for all $X \in (a,b)$,

$$u \in L_{p'}(a,X) \text{ and } v \in L_p(X,b). \tag{6.2}$$

Under these restrictions on u and v it is well known (see Theorem 4.1) that the norm $\|T\|$ of the operator $T : L_p(a,b) \to L_p(a,b)$ satisfies

$$\|T\| \sim \sup_{x\in(a,b)} \|u\chi_{(a,x)}\|_{p',(a,b)}\|v\chi_{(x,b)}\|_{p,(a,b)}, \tag{6.3}$$

the constants implicit in the symbol $\sim$ being absolute ones independent of u, v and (a,b). Here χ_S denotes the characteristic function of the set S and, as before,

$$\|f\|_{p,I} = \left(\int_I |f(t)|^p dt\right)^{1/p}, \quad 1 \le p < \infty, \quad I \subset (a,b);$$

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$\|f\|_{\infty,I}$ is defined in the obvious way. For given interval $I=(c,d)\subset(a,b)$, define

$$\mathscr{J}(I)=\sup_{x\in I}\|u\chi_{(c,x)}\|_{p',I}\|v\chi_{(x,d)}\|_{p,I} \tag{6.4}$$

and

$$(T_{e,I}f)(x):=v(x)\chi_I(x)\int_e^x u(t)\chi_I(t)f(t)dt,\ \text{where } e\in[a,b]. \tag{6.5}$$

Then the norm of the operator $T_{a,I}:L_p(I)\to L_p(I)$ satisfies

$$\|T_{a,I}\|\sim\mathscr{J}(I), \tag{6.6}$$

with absolute constants independents of p and I. Let us recall that from Theorem 4.4 it follows that

$$u\in L_{p'}(a,b) \text{ and } v\in L_p(a,b) \tag{6.7}$$

guarantee compactness of T.

Lemma 6.1. *Suppose that $1<p<\infty$. Then the function $\mathscr{J}(.,d)$ is continuous and non-increasing on (a,d), for any $d<b$, and $\mathscr{J}(e,.)$ is continuous and non-increasing on (a,b) for any $e>a$.*

Proof. Given $x\in(a,b)$ and $\varepsilon>0$, there exists $h=h(x,\varepsilon)\in(0,\min\{\frac{1}{2}(x+a),b-x\})$ such that

$$\left(\int_{x-h}^{x+h}|u(t)|^{p'}dt\right)^{1/p'}<\min\left(\frac{\varepsilon}{\|v\|_{p,(\frac{x-a}{2},d)}+1},\varepsilon\right).$$

Then

$$\begin{aligned}J^{p'}(x,d)\le J^{p'}(x-h,d)&=\max\left\{\sup_{x-h<z<x}\left[\int_{x-h}^{z}|u(t)|^{p'}dt\ \|v\|_{p,(z,d)}^{p'}\right],\right.\\&\left.\sup_{x<z<d}\left[\left(\int_{x-h}^{x}+\int_x^z\right)|u(t)|^{p'}dt\ \|v\|_{p,(z,d)}^{p'}\right]\right\}\\&\le\max\{\varepsilon^{p'},\varepsilon^{p'}+[\mathscr{J}(x,d)]^{p'}\}=\varepsilon^{p'}+[\mathscr{J}(x,d)]^{p'}\end{aligned}$$

and so $0<[\mathscr{J}(x-h,d)]^{p'}-[\mathscr{J}(x,d)]^{p'}<\varepsilon$. Similarly $0<[\mathscr{J}(x)]^{p'}-[\mathscr{J}(x+h)]^{p'}<\varepsilon$ and the continuity is established. It is obvious that $\mathscr{J}(.,d)$ is non-increasing. For the function $\mathscr{J}(e,.)$ the proof is similar and hence the lemma is proved. □

Next we introduce a function $\mathscr{A}$ which will play a key role in this section.

Definition 6.1. Let I be a subset of (a,b). We define

$$\mathscr{A}(I)\equiv\mathscr{A}(I,u,v):=\begin{cases}\sup_{f\in L_p(I),f\neq0}\inf_{\alpha\in\mathbb{C}}\frac{\|T_{a,I}f-\alpha v\ \|_{p,I}}{\|f\|_{p,I}} & \text{if } \mu(I)>0,\\ 0 & \text{if } \mu(I)=0,\end{cases}$$

where $T_{a,I}f(x)$ is as in (6.1) and

$$\mu(I) := \begin{cases} \int_I |v(t)|^p dt, & 1 \le p < \infty, \\ \int_I |v(t)| dt, & p = \infty. \end{cases}$$

6.2 Properties of $\mathscr{A}$

In this section we establish properties of the function $\mathscr{A}$ which we shall need in the next sections.

With help of (4.5) we have for $I = (a,b)$,

$$\begin{aligned} \mathscr{A}(I) &\le \sup_{f \in L_p(I), f \ne 0} \frac{\|T_{a,I}f\|_{p,I}}{\|f\|_{p,I}} \\ &= \|T_{a,I}\| \le 4\mathscr{J}(I). \end{aligned} \tag{6.8}$$

Note that from Definition 6.1 the next corollary immediately follows.

Corollary 6.1. *For all subintervals $I \subset (a,b)$,*

$$\mathscr{A}(I) \le \inf_{y \in I} \|T_{y,I}|L_p(I) \to L_p(I)\|.$$

Lemma 6.2. *Let $1 \le p \le \infty$ and $I = (c,d) \subset (a,b)$. Then $\|T_{x,I}|L_p(I) \to L_p(I)\|$ is continuous in x.*

Proof. Let $I = (c,d) \subset (a,b)$, $x \in (a,b)$ and e be the nearest point of $\bar{I}$ to x. Then $T_{x,I} = T_{e,I}$ and $\|T_{e,I}\| := \|T_{e,I}|L_p(I) \to L_p(I)\| \le \|T_x\| := \|T_x|L_p(a,b) \to L_p(a,b)\|$. Moreover if $I' \subset I$, with e' the nearest point of $\bar{I}'$ to x then $T_{e,I}f = T_{e,I}(f\chi_{(e,e')}) + T_{e',I'}(f\chi_{I'})$, whence $0 \le \|T_{e,I}\| - \|T_{e',I'}\| \le \|u\|_{p',(e,e')}\|v\|_{p,(c,b)}$. This yields the lemma. □

Lemma 6.3. *Suppose T_a is bounded and $e \in (c,d) = I \subseteq (a,b)$. Then*

$$\min\{\|T_{e,(c,e)}\|, \|T_{e,(e,d)}\|\} \le \min_{x \in I} \|T_{x,I}\|.$$

Proof. We can see that for $x \in I$, $\|T_{x,I}\| = \max\{\|T_{x,(c,x)}\|, \|T_{x,(x,d)}\|\}$ and with the help of Lemma 6.2 the lemma follows. □

In the next two lemmas $\|\cdot\|_{p,\mu}$ denotes the norm in $L_p(I,d\mu)$, where $d\mu(t) = |v(t)|^p dt$.

Lemma 6.4. *If $1 < p \le \infty$, then given any $f, e_0 \in L_p(I,d\mu)$ with $e_0 \ne 0$ there is a unique scalar $c_f = c_{f,e_0}$ such that $\|f - c_f e_0\|_{p,\mu} = \inf_{c \in \mathbb{C}} \|f - ce_0\|_{p,\mu}$.*

Proof. Since $\|f-ce_0\|_{p,\mu}$ is continuous in c and tends to ∞ as $c\to\infty$, the existence of c_{f,e_0} is guaranteed by the local compactness of $\mathbb{C}$. For $1<p<\infty$ the uniqueness follows from the uniform convexity of $L_p(I,d\mu)$. Let $p=\infty$, and suppose that there are two distinct values c_1,c_2 of c_f. This yields the contradiction $\|f-(1/2)(c_1+c_2)e_0\|_{p,\mu}<\|f-c_1e_0\|_{p,\mu}$. □

Lemma 6.5. *Let $e_0\in L_p(I,d\mu)\setminus\{0\}$ and let $c_f=c_{f,e_0}$ be as in Lemma 6.4. The map $f\mapsto c_f : L_p(I,d\mu)\mapsto\mathbb{C}$ is continuous for $1<p\le\infty$.*

Proof. Suppose that $g_n\to f$. Since $\{c_{g_n}\}$ is bounded we may suppose that $c_{g_n}\to c$ as $g_n\to f$. Then

$$\|g_n-c_fe_0\|_{p,\mu}\ge\|g_n-c_{g_n}e_0\|_{p,\mu}$$

and so

$$\|f-c_fe_0\|_{p,\mu}\ge\|f-ce_0\|_{p,\mu}$$

which gives $c=c_f$. □

Theorem 6.1. *Let $1<p<\infty$ and $I=(c,d)\subseteq(a,b)$. Then there exists $e\in I$ such that*

$$\mathscr{A}(I)=\min_{x\in I}\|T_{x,I}|L_p(I)\to L_p(I)\|=\|T_{e,I}|L_p(I)\to L_p(I)\|$$

Proof. Since $\|T_{x,(c,x)}\|$ and $\|T_{x,(x,d)}\|$ are monotone in x we can see that there is $y\in I$ at which $\|T_{y,(c,y)}\|=\|T_{y,(y,d)}\|$. Then using Lemma 6.3 we get $\|T_{y,I}\|=\min_{x\in I}\|T_{x,I}\|$. If $\alpha<\|T_{y,I}\|$ there exist $f_i, i=1,2$, supported in (c,y) and (y,d), respectively, with $\|f_i\|=1, \|T_{b,I}f_i\|>\alpha$, and f_1 positive, f_2 negative. Clearly the corresponding values of c_f, say c_1,c_2, are positive and negative respectively. Then by continuity there is a $\lambda\in[0,1]$ such that $c_g=0$ for $g=\lambda f_1+(1-\lambda)f_2$, and $\|T_{y,I}g\|^p=\lambda^p\|T_{y,I}f_1\|^p+(1-\lambda)^p\|T_{y,I}f_2\|^p>\alpha^p\|g\|^p$. Then, by Lemma 6.4,

$$\mathscr{A}(I)\ge\inf_c\|(T_{y,I}-cv)g\|/\|g\|=\|T_{b,I}g\|/\|g\|>\alpha.$$

Since $\alpha<\|T_{y,I}\|$ is arbitrary, $\mathscr{A}(I)\ge\|T_{y,I}\|$ and the first equality follows from Corollary 6.1. Using the obvious facts that $\|T_{c,I}|L_p(I)\to L_p(I)\|$ and $\|T_{d,I}|L_p(I)\to L_p(I)\|$ are greater than $\|T_{x,I}|L_p(I)\to L_p(I)\|$ for any $x\in(c,d)$, and the continuity of $\|T_{x,I}|L_p(I)\to L_p(I)\|$ in the variable x, we obtain the second equality. □

Lemma 6.6. *Suppose that $a\le c<d\le b$, $e\in(a,b)$ and $1<p<\infty$. Then:*

1. *The function $\mathscr{A}(.,d)$ is non-increasing and continuous on (a,d).*
2. *The function $\mathscr{A}(c,.)$ is non-decreasing and continuous on (c,b).*
3. $\lim_{y\to e_+}\mathscr{A}(e,y)=\lim_{y\to e_-}\mathscr{A}(y,e)=0.$

Proof. The proof of 1 illustrates the techniques necessary to prove 2 and 3 also. That $\mathscr{A}(.,d)$ is non-increasing is easy to see. To establish continuity from the left, fix $y\in(a,d)$. Then there exists $h_0>0$ such that for $0<h<h_0$,

$$
\begin{aligned}
&\mathscr{A}^p(y,d) \le \mathscr{A}^p(y-h,d) \\
&= \sup_{\|f\|_{p,(y-h,d)}\le 1} \inf_{|\alpha|\le\|u\|_{p',(y-h_0,d)}} \|v\left[\int_{y-h}^{\cdot} u(t)f(t)dt-\alpha\right]\|^p_{p,(y-h,d)} \\
&= \sup_{\|f\|_{p,(y-h,d)}\le 1} \inf_{|\alpha|\le\|u\|_{p',(y-h_0,d)}} \left[\|v\left[\int_{y-h}^{y} u(t)f(t)dt-\alpha\right]\|^p_{p,(y-h,y)}\right. \\
&\quad \left. + \|v\left[\int_{y}^{\cdot} u(t)f(t)dt-\alpha+\int_{y-h}^{y} u(t)f(t)dt\right]\|^p_{p,(y,d)}\right] \\
&\le \sup_{\|f\|_{p,(y-h,d)}\le 1} \inf_{|\alpha|\le\|u\|_{p',(y-h_0,d)}} \left[2\|v\int_{y-h}^{\cdot} u(t)f(t)dt\|^p_{p,(y-h,y)}\right. \\
&\qquad \left. +2\alpha^p\|v\|^p_{p,(y-h,y)} + \|v\left[\int_{y}^{\cdot} u(t)f(t)dt-\alpha+\int_{y-h}^{y} u(t)f(t)dt\right]\|^p_{p,(y,d)}\right] \\
&\le 2\|u\|^p_{p',(y-h,y)}\|v\|^p_{p,(y-h,y)} + 2\|u\|^p_{p',(y-h_0,d)}\|v\|^p_{p,(y-h,y)} \\
&\qquad +2\|u\|^p_{p',(y-h,y)}\mathscr{A}^p(y,d) + \mathscr{A}^p(y,d).
\end{aligned}
$$

It follows that

$$\lim_{h\to 0_+} \mathscr{A}(y-h,d) = \mathscr{A}(y,d).$$

In the same way we see that

$$\lim_{h\to 0_+} \mathscr{A}(y+h,d) = \mathscr{A}(y,d),$$

and now the proof of the Lemma is complete. □

Note that when $p=\infty$ or $p=1$ then A can be discontinuous as we can see from the following example. Set

$$
\begin{aligned}
v(x) &= \begin{cases} 1, & x\in(0,1)\cup(2,\infty), \\ \varepsilon, & \text{otherwise}, \end{cases} \\
u(x) &= \chi_{(1,2)}(x) + \varepsilon\chi_{(0,1)\cup(2,3)}.
\end{aligned}
$$

with $(a,b)=(0,3)$. Then $\mathscr{A}(x,3)<\varepsilon$ for $x>1$, and $\mathscr{A}(x,3)>1/2$ for $x<1$.

Lemma 6.7. *Suppose that $T: L_p(a,b)\to L_p(a,b)$ is bounded and $1<p<\infty$. Let $I=(c,d)$ and $J=(c',d')$ be subintervals of (a,b), with $J\subset I$, $|J|>0$, $|I-J|>0$, $\int_a^b v^p(x)dx<\infty$ and $u,v\neq 0$ a.e. on I. Then*

$$\mathscr{A}(I) > \mathscr{A}(J) > 0. \tag{6.9}$$

and

$$\|T_{a,I}\| > \|T_{a,J}\| > 0. \tag{6.10}$$

Proof. Since $|\{x : u(x) = 0\}| = |\{x : v(x) = 0\}| = 0$ the proof of (6.10) is obvious.

Let $0 \le f \in L_p(J)$, $0 < \|f\|_{p,J} = \|f\|_{p,I} \le 1$ with $\operatorname{supp} f \subset J$. Let $y \in J$; then

$$\|T_{(c',y)}\|_{p,J} > 0 \qquad \text{and} \qquad \|T_{(y,d')}\|_{p,J} > 0$$

and then from Lemma 6.2 we have

$$\min\{\|T_{(c',y)}\|_{p,J}, \|T_{(y,d')}\|_{p,J}\} \le \min_{x\in J} \|T_{x,J}\|_{p,J}$$

which means $\mathscr{A}(J) > 0$.

Next, let us suppose that $c = c' < d' < d$. By Theorem 6.1, there exist $x_0 \in J$ and $x_1 \in I$ such that $\mathscr{A}(J) = \|T_{x_0,J}\|_{p,J}$ and $\mathscr{A}(I) = \|T_{x_1,I}\|_{p,I}$.

If $x_0 = x_1$, then, since $u, v \ne 0$ a.e. on I, we get

$$\mathscr{A}(I) = \|T_{x_1,I}\|_{p,I} > \|T_{x_1,I}\|_{p,J} = \|T_{x_1,J}\|_{p,J} = \mathscr{A}(J).$$

If $x_0 \ne x_1$, then

$$\mathscr{A}(I) = \|T_{x_1,I}\|_{p,I} \ge \|T_{x_1,I}\|_{p,J} \ge \|T_{x_1,J}\|_{p,J} > \|T_{x_0,J}\|_{p,J} = \mathscr{A}(J).$$

The case $c < c' < d' = d$ can be proved similarly; the result when $c < c' < d' < d$ follows from previous cases and the monotonicity of $\mathscr{A}(I)$. □

Lemma 6.8. *Let $1 < p < \infty$, let u, v be constants over a finite real interval $I = (a,b)$ and put $d = (a+b)/2$. Then*

$$\begin{aligned}\mathscr{A}(I,u,v) &= |v||u||I|\mathscr{A}((0,1),1,1)\\ &= \sup_{f\in L^p(I)\setminus\{0\}} \frac{\|v(x)\int_d^x u(t)f(t)dt\|_{p,I}}{\|f\|_{p,I}}\\ &= |u||v|\frac{\|\sin_p(\pi_p(x-a)/(b-a))\|_{p,I}}{\|\cos_p(\pi_p(x-a)/(b-a))\|_{p,I}}\\ &= |u||v|\gamma_p,\end{aligned}$$

where γ_p is as in Theorem 5.8.

Proof. We have

$$\begin{aligned}\mathscr{A}(I,u,v) &= \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\mathbb{C}} \|v\left(\int_a^{\cdot} u(t)f(t)dt - \alpha\right)\|_{p,I}\\ &= |v||u| \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\mathbb{C}} \|\int_a^{\cdot} f(t)dt - \alpha\|_{p,I}\end{aligned}$$

$$= |v||u||I| \sup_{\|f\|_{p,(0,1)}\le 1} \inf_{\alpha\in\mathbb{C}} \|\int_0^{\cdot} f(t)dt - \alpha\|_{p,(0,1)}$$
$$= |v||u||I|\mathscr{A}((0,1),1,1).$$

The rest follows from Remark 5.2 and Theorem 5.8. □

Next, we investigate the dependence of $\mathscr{A}(I,u,v)$ on u and v.

Lemma 6.9. *Let $I=(c,d)\subset(a,b)$, $1\le p\le\infty$, and suppose that $v\in L_p(I)$ and $u_1,u_2\in L_{p'}(I)$. Then*

$$|\mathscr{A}(I,u_1,v)-\mathscr{A}(I,u_2,v)|\le \|u_1-u_2\|_{p',I}\|v\|_{p,I}.$$

Proof. Without loss of generality we may suppose that $\mathscr{A}(I,u_1,v)\ge\mathscr{A}(I,u_2,v)$. Then

$$\begin{aligned}
\mathscr{A}(I,u_1,v) &= \sup_{\|f\|_{p,I}\le 1}\inf_{\alpha\in\mathbb{R}} \|v\left[\int_c^{\bullet}(u_1-u_2+u_2)f dt-\alpha\right]\|_{p,I}\\
&\le \sup_{\|f\|_{p,I}\le 1}\inf_{\alpha\in\mathbb{R}}\Big[\|v\int_c^{\bullet}(u_1-u_2)f dt\|_{p,I}\\
&\qquad + \|v(\int_c^{\bullet}u_2 f dt-\alpha)\|_{p,I}\Big]\\
&\le \sup_{\|f\|_{p,I}\le 1}\inf_{\alpha\in\mathbb{R}}\Big[\|v\|_{p,I}\|u_1-u_2\|_{p',I}\\
&\qquad + \|v(\int_c^{\bullet}u_2 f-\alpha)\|_{p,I}\Big]\\
&\le \|v\|_{p,I}\|u_1-u_2\|_{p',I}+\mathscr{A}(I,u_2,v).
\end{aligned}$$

The result follows. □

Lemma 6.10. *Let $I=(c,d)\subset(a,b)$, $1\le p\le\infty$, and suppose that $u\in L_{p'}(I)$ and $v_1,v_2\in L_p(I)$. Then*

$$|\mathscr{A}(I,u,v_1)-\mathscr{A}(I,u,v_2)|\le \|u\|_{p',I}\|v_1-v_2\|_{p,I}.$$

Proof. We may suppose that $\mathscr{A}(I,u,v_1)\ge\mathscr{A}(I,u,v_2)$. Then

$$\begin{aligned}
\mathscr{A}(I,u,v_1) &= \sup_{\|f\|_{p,I}\le 1}\inf_{\alpha\in\mathbb{R}} \|v_1\left[\int_c^{\bullet}uf dt-\alpha\right]\|_{p,I}\\
&= \sup_{\|f\|_{p,I}\le 1}\inf_{|\alpha|\le\|u\|_{p',I}\|f\|_{p,I}} \|v_1\left[\int_c^{\bullet}uf dt-\alpha\right]\|_{p,I}
\end{aligned}$$

$$\le \sup_{\|f\|_{p,I}\le 1} \inf_{|\alpha|\le\|u\|_{p',I}} \Big[\|(v_1-v_2)[\int_c^{\cdot} uf dt - \alpha]\|_{p,I}$$
$$+\|v_2[\int_c^{\cdot} uf dt - \alpha]\|_{p,I}\Big]$$
$$\le \|v_1-v_2\|_{p,I}\|u\|_{p',I} + \mathscr{A}(I,u,v_2).$$

The proof is complete. □

Remark 6.1. Note that in Lemmas 6.6–6.10 we can replace $\mathscr{A}(c,d)$ by $\|T_{a,(c,d)}\|_{p,(c,d)}$.

6.3 Equivalence of Strict *s*-Numbers for *T*

Throughout this section we suppose that $1 < p < \infty$ if not mentioned otherwise.

Remark 6.2. Let T be compact. Then it follows from the continuity of $\mathscr{A}(.,b)$ and $\|T_{a,(a,.)}\|$ and Theorem 4.4 that for sufficiently small $\varepsilon > 0$ there are $c,d \in (a,b)$ for which $\mathscr{A}(c,b) = \varepsilon$ and $\|T_{a,(a,d)}|L_p(a,d) \to L_p(a,d)\| = \varepsilon$. Indeed, since T is compact, there exists a positive integer $N(\varepsilon)$ and points $a = a_0 < a_1 < \cdots < a_{N(\varepsilon)} = b$ with $\mathscr{A}(a_{i-1},a_i) = \varepsilon$ for $i = 2,\dots,N(\varepsilon)-1$, $\mathscr{A}(a_{n-1},b) \le \varepsilon$ and $\|T_{a,(a,a_1)}|L_p(a,a_1) \to L_p(a,a_1)\| = \varepsilon$. Clearly, the intervals $I_i = (a_{i-1},a_i), i = 1,\dots,N(\varepsilon)$ form a partition of (a,b).

Lemma 6.11. *If $T : L_p(a,b) \to L_p(a,b)$ is compact and v,u satisfy (6.2), then the number $N(\varepsilon)$ is a non-increasing function of ε which takes on every sufficiently large integer value.*

Proof. Fix $c, a < c < b$. Then, continuity of $\|T_{a,(a,.)}\|$ with Theorem 4.4 and (4.5) ensures $\|T_{a,(a,c)}\| = \varepsilon_0 > 0$. Moreover, as observed in Remark 6.2 there is a positive integer $N(\varepsilon_0)$ and a partition $a = a_0 < a_1 = c < \cdots < a_{N(\varepsilon_0)} = b$ such that $\|T_{a,(a_0,a_1)}\| = \varepsilon_0$, $\mathscr{A}(a_i,a_{i+1}) = \varepsilon_0$, $i = 1,2,\dots,N(\varepsilon_0)-1$ and $\mathscr{A}(a_{N(\varepsilon_0)-1},b) \le \varepsilon_0$. Let $d \in (a,c)$. According to Lemma 6.7, $\mathscr{A}(a,d) = \varepsilon_0' < \varepsilon_0$ and the procedure outlined above applied with ε_0' gives $\infty > N(\varepsilon_0') \ge N(\varepsilon_0)$. (Note that due to Lemma 6.7, the compactness of T and the continuity of $\mathscr{A}(c,.)$ and $\|T_{a,(a,.)}\|$, there exists $d \in (a,c)$ such that $N(\varepsilon_0') > N(\varepsilon_0)$.) If $N(\varepsilon_0') = N(\varepsilon_0)+1$, stop.

Otherwise, define

$$\varepsilon_1 = \sup\{\varepsilon : 0 < \varepsilon < \varepsilon_0 \text{ and } N(\varepsilon) \ge N(\varepsilon_0)+1\}.$$

We claim $N(\varepsilon_1) = N(\varepsilon_0)+1$. Indeed, suppose $N(\varepsilon_1) \ge N(\varepsilon_0)+2$ and the partition $a = a_0 < a_1 < \cdots < a_{N(\varepsilon_1)} = b$ satisfies $\|T_{a,(a,a_1)}\| = \varepsilon_1$, $\mathscr{A}(a_i,a_{i+1}) = \varepsilon_1$, $i = 1,2,\dots,N(\varepsilon_1)-2$ and $\mathscr{A}(a_{N(\varepsilon_1)-1},a_{N(\varepsilon_1)}) \le \varepsilon_1$. Decrease $a_{N(\varepsilon_1)-1}$ slightly to $a'_{N(\varepsilon_1)-1}$ so that both $\mathscr{A}(a'_{N(\varepsilon_1)-1},b) < \varepsilon_1$ and $\mathscr{A}(a_{N(\varepsilon_1)-2},a'_{N(\varepsilon_1)-1}) > \varepsilon_1$, continuing the process to get a partition of (a,b) having $N(\varepsilon_1)$ intervals such that

$\|T_{a,(a,a_1')}\| > \varepsilon_1$, $\mathscr{A}(a_{i-1}', a_i') > \varepsilon_1, i = 2, \ldots, N(\varepsilon_1) - 1$ and $\mathscr{A}(a_{N(\varepsilon_1)-1}', b) < \varepsilon_1$. Taking $\varepsilon_2 \leq \min\{\|T_{a,(a,a_1')}\|, \mathscr{A}(a_{i-1}', a_i') : 2 \leq i \leq N(\varepsilon_1) - 1\}$ we obtain $\varepsilon_2 > \varepsilon_1$ and $N(\varepsilon_2) \geq N(\varepsilon_0) + 2$, a contradiction. This establishes the claim.

An inductive argument completes the proof. □

Lemma 6.12. *Let $\varepsilon > 0$, $1 < p < \infty$ and let $T : L_p(a,b) \to L_p(a,b)$ be compact, with $u \in L_{p'}(a,b)$, $v \in L_p(a,b)$. Let $a = a_0 < a_1 < \ldots < a_N = b$ be a sequence such that $\mathscr{A}(a_{i-1}, a_i) \leq \varepsilon$ for $i = 2, \ldots, N$ and $\|T_{a,(a_0,a_1)}\| \leq \varepsilon$. Then*

$$a_N(T) \leq \varepsilon.$$

Proof. Set $I_i = (a_{i-1}, a_i)$ and $Pf = \sum_{i=2}^{N} P_i f + \chi_{I_1}$ where

$$P_i f(x) := \chi_{I_i}(x) v(x) \left[\int_a^{e_i} u(t) f(t) dt \right],$$

and $e_i \in I_i$ is a number obtained from Theorem 6.1 for which

$$\mathscr{A}(I_i) = \min_{x \in I_i} \|T_{x,I_i} | L_p(I_i) \to L_p(I_i)\| = \|T_{e_i,I_i} | L_p(I_i) \to L_p(I_i)\|. \tag{6.11}$$

Then $\operatorname{rank} P \leq N - 1$ and, on using Theorem 6.1,

$$\begin{aligned}
\|(T-P)f\|_{p,(a,b)}^p &= \sum_{i=2}^{N} \|(Tf - Pf)\|_{p,I_i}^p + \|Tf\|_{p,I_1}^p \\
&= \sum_{i=2}^{N} \|Tf - P_i f\|_{p,I_i}^p + \|Tf\|_{p,I_1}^p \\
&= \sum_{i=2}^{N} \|v(.) \int_{e_i}^{.} u(t) f(t) dt\|_{p,I_i}^p + \|v(.) \int_a^{.} u(t) f(t) dt\|_{p,I_1}^p \\
&\leq (\max\{\|T\|_{p,I_1}, \mathscr{A}(I_2), \ldots, \mathscr{A}(I_N)\})^p \|f\|_{p,(a,b)}^p \\
&\leq \varepsilon^p \|f\|_{p,(a,b)}^p,
\end{aligned}$$

whence the lemma. □

Lemma 6.13. *Let $\varepsilon > 0$, $1 < p < \infty$ and let $T : L_p(a,b) \to L_p(a,b)$ be compact and $v \in L_p(a,b)$, $u \in L_{p'}(a,b)$. Let $a = a_0 < a_1 < \ldots < a_N = b$ be a sequence such that $\mathscr{A}(a_{i-1}, a_i) \geq \varepsilon$ for $i = 2, \ldots, n$ and $\|T_{a,(a_0,a_1)}\| \geq \varepsilon$. Then*

$$i_n(T) \geq \varepsilon.$$

Proof. Set $I_i = (a_{i-1}, a_i)$; then from Theorem 6.1 it follows that there is $e_i \in I_i$ such that

$$\mathscr{A}(I_i) = \min_{x \in I_i} \|T_{x,I_i} | L_p(I_i) \to L_p(I_i)\| = \|T_{e_i,I_i} | L_p(I_i) \to L_p(I_i)\|. \tag{6.12}$$

Since T is compact there exist functions f_i such that $\text{supp} f_i \subseteq I_i$, $\|f_i\|_{p,I_i} = 1$ and $\|T_{e_i} f_i\|_{p,I_i} = \mathscr{A}(I_i)$ for $i = 1,\dots,n-1$.

Define $J_1 = (a_0,e_1) = (e_0,e_1)$, $J_i = (e_{i-1},e_i)$ for $i = 2,\dots,n-1$ and $J_n = (e_{n-1},b) = (e_{n-1},e_n)$. We introduce functions $g_i(x) = (c_i f_i(x) + d_i f_{i+1}(x))\chi_{J_i}(x)$ for $i = 1,\dots,n-1$ and $g_n(x) = c_n f_n(x)\chi_{J_n}(x)$. For these functions we have

$$\frac{\|T_{e_{i-1}} g_i\|_{p,(e_{i-1},a_{i-1})}}{\|g_i\|_{p,(e_{i-1},a_{i-1})}} \geq \varepsilon \text{ and } \frac{\|T_{e_i} g_i\|_{p,(a_{i-1},e_i)}}{\|g_i\|_{p,(a_{i-1},e_i)}} \geq \varepsilon \text{ for } i = 1,\dots,n-1.$$

Also we can see that $T_{e_{i-1},(e_{i-1},a_{i-1}),1,u} g_i$ and $T_{e_i,(a_{i-1},e_i),1,u} g_i$ do not change sign on (e_{i-1},a_{i-1}) and (a_{i-1},e_i) respectively. Since $T_{e_{i-1},(e_{i-1},a_{i-1}),1,u} g_i(.)$ and $T_{e_i,(a_{i-1},e_i),1,u}$ $g_i(.)$ are continuous functions we can choose constants c_i and d_i such that

$$T_{e_{i-1},(e_{i-1},a_{i-1}),1,u} g_i(a_{i-1}) = T_{e_i,(a_{i-1},e_i),1,u} g_i(a_{i-1}) > 0$$

and $\|g_i\|_{p,J_i} = 1$. Then we can see that $\text{supp}(Tg_i) \subseteq J_i$.

Note that we have

$$\frac{\|Tg_i\|_{p,J_i}}{\|g_i\|_{p,J_i}} \geq \varepsilon \text{ for } i = 1,\dots,n. \tag{6.13}$$

The maps $A : l_p^n \to L_p(a,b)$ and $B : L_p(a,b) \to l_p^n$ are defined by:

$$\mathscr{A}(\{d_i\}_{i=1}^n) = \sum_{i=1}^n d_i g_i(x)$$

$$Bg(x) = \left\{ \frac{\int_{J_i} g(x)(Tg_i(x))_{(p)}dx}{\|Tg_i(x)\|_{p,J_i}^p} \right\}_{i=1}^n .$$

Since $\int_I f(f)_{(p)}dx = \|f\|_{p,I}^p = 1$,

$$\begin{aligned} BT(\mathscr{A}(\{d_i\}_{i=1}^n)) &= \left\{ \left(\int_{J_i} d_i Tg_i(x)(Tg_i)_{(p)}dx \right) / \|Tg_i\|_{p,J_i}^p \right\}_{i=1}^n \\ &= \left\{ d_i \|Tg_i\|_{p,J_i}^p / \|Tg_i\|_{p,J_i}^p \right\}_{i=1}^n = \{d_i\}_{i=1}^n . \end{aligned}$$

Observe that $\|B : L_p(a,b) \to l_p^n\|$ is attained only for functions of the form:

$$g(x) = \sum_{i=1}^n c_i' Tg_i(x).$$

Using (6.13) we obtain

$$\|g\|_{p,(a,b)} \geq \varepsilon \|\{c_i'\}_{i=1}^n\|_{l_p^n}$$

and then

$$\sup_{\|f\|_{L^p(a,b)} \leq 1} \|Bf\|_{l_p^n} = \sup_{\|g\|_{L^p(a,b)} \leq 1} \|B(\sum_{i=1}^n c_i' Tg_i(x))\|_{l_p^n} = \sup_{\|g\|_{L^p(a,b)} \leq 1} \|c_i'\|_{l_p^n} \leq \frac{1}{\varepsilon}.$$

From

$$\|\mathscr{A}(\{d_i'\}_{i=1}^n)\|_{p,(a,b)}^p = \sum_{i=1}^n \int_{J_i} |d_i' g_i(x)|^p dx = \sum_{i=1}^n |d_i'|^p \|g_i(x)\|_{p,J_i}^p dx = \|d_i'\|_{l_p^n}^p$$

it follows that $\|A : l_p^n \to L_p(a,b)\| = 1$. Thus

$$i_n(T) \geq \|A\|^{-1}\|B\|^{-1} \geq \varepsilon.$$

□

From Lemma 6.7, Remark 6.2, Lemma 6.11 and continuity of $\mathscr{A}(c.,)$ and $\|T_{a,(c,.)}\|_{p,(c,.)}$ the next lemma follows.

Lemma 6.14. *If $T : L_p(a,b) \to L_p(a,b)$ is compact, then for each $N > 1$ there exist $\varepsilon_N > 0$ and a sequence $a = a_0 < a_1 < \ldots < a_N = b$ such that $\mathscr{A}(a_{i-1}, a_i) = \varepsilon_N$ for $i = 2, \ldots, N$ and $\|T_{a,(a_0,a_1)}\| = \varepsilon_N$.*

Combining Lemmas 6.12–6.14 we obtain the following theorem.

Theorem 6.2. *Let $1 < p < \infty$ and let $T : L_p(a,b) \to L_p(a,b)$ be compact; let $N > 1$. Then there exist $\varepsilon_N > 0$ and a sequence $a = a_0 < a_1 < \ldots < a_N = b$ such that $\mathscr{A}(a_{i-1}, a_i) = \varepsilon_N$ for $i = 2, \ldots, N$, $\|T_{a,(a_0,a_1)}\| = \varepsilon_N$ and*

$$a_N(T) = i_N(T) = \varepsilon_N.$$

An obvious consequence of the above theorem is

Remark 6.3. Let $1 < p < \infty$ and let $T : L_p(a,b) \to L_p(a,b)$ be compact. Then all strict s-numbers for the map T coincide.

6.4 The First Asymptotic Term when $1 < p < \infty$

Theorem 6.3. *Let $1 < p < \infty$, $v \in L_p(a,b)$, $u \in L_{p'}(a,b)$ and let $T : L_p(a,b) \to L_p(a,b)$ be compact. Then*

$$\gamma_p \int_a^b |u(t)v(t)|dt = \lim_{N\to\infty} \varepsilon_N N$$

where γ_p is as in Theorem 5.8 and ε_N as in Lemma 6.14.

Proof. For each $\eta > 0$ there exist step functions u_η, v_η on $I = (a,b)$ such that

$$\|u - u_\eta\|_{p',I} < \eta$$
$$\|v - v_\eta\|_{p,I} < \eta.$$

We may assume that

$$u_\eta = \sum_{j=1}^{m} \xi_j \chi_{W(j)}, \qquad v_\eta = \sum_{j=1}^{m} \eta_j \chi_{W(j)},$$

where the $W(j)$ are disjoint subintervals of I.

Let N be an integer greater than 1. Then according to Lemma 6.14 there exist $\varepsilon_N > 0$ and a sequence $a_k, k = 0, 1, \ldots, N$, such that $a_0 = a$, $a_N = b$ and

$$\mathscr{A}(I_i) = \varepsilon \text{ for } i = 2, \ldots, N \text{ and } \|T_{a,a_1}\| = \varepsilon \text{ where } I_k = [a_{k-1}, a_k].$$

Then

$$\begin{aligned} &\left| \int_I |u(t)\, v(t)|dt - \int_I |u_\eta(t) v_\eta(t)|dt \right| \\ &\qquad \le \int_I |u(t)| \left| v_\eta(t) - v(t) \right| dt + \int_I |u(t) - u_\eta(t)|\, |v_\eta(t)| dt \\ &\qquad < 2\eta (\|u\|_{p',I} + \|v\|_{p,I}). \end{aligned} \tag{6.14}$$

Next, let $\mathbb{K} := \{k > 1 : \text{there exists } j \text{ such that } I_k \subset W(j)\}$. Then $\#\mathbb{K} \ge N - 1 - m$, and, by Lemmas 6.8–6.10,

$$\begin{aligned} (N - 1 - m)\varepsilon &\le \sum_{k \in \mathbb{K}} \mathscr{A}(I_k; u, v) \\ &\le \sum_{k \in \mathbb{K}} \big\{ \mathscr{A}(I_k; u_\eta, v_\eta) \\ &\qquad + (\mathscr{A}(I_k; u, v) - \mathscr{A}(I_k; u_\eta, v)) \\ &\qquad + (\mathscr{A}(I_k; u_\eta, v) - \mathscr{A}(I_k; u_\eta, v_\eta)) \big\} \\ &\le \gamma_p \sum_j |\xi_j||\eta_j||W(j)| \\ &\qquad + \sum_j \big\{ \|u - u_\eta\|_{p',W(j)} \|v\|_{p,W(j)} \\ &\qquad\qquad + \|u_\eta\|_{p',W(j)} \|v_\eta - v\|_{p,W(j)} \big\} \\ &\le \gamma_p \int_I |u_\eta||v_\eta| dt + \|u - u_\eta\|_{p',I} \|v\|_{p,I} \\ &\qquad + \|u_\eta\|_{p',I} \|v_\eta - v\|_{p,I} \\ &\le \gamma_p \int_I |u_\eta||v_\eta| dt + \eta \|v\|_{p,I} + \eta (\|u\|_{p',I} + \eta). \end{aligned}$$

By (6.14) we therefore conclude that

$$\limsup_{N \to \infty} \varepsilon_N N \le \int_I |u(t)||v(t)| dt + 3\eta \|v\|_{p,I} + \eta (3\|u\|_{p',I} + \eta)$$

and then

$$\limsup_{N\to\infty} \varepsilon_N N \leq \int_I |u(t)||v(t)|dt.$$

To prove the opposite inequality we add the end-points of the intervals $W(j)$, $j = 1,2\ldots,m$ to the $a_k, k = 0,1,\ldots,N$, to form the partition $a = e_0 < \cdots < e_n = b$, say, where $n \leq N+1+m$. Note that each interval $J_i := [e_i, e_{i+1}]$ is a subinterval of some $W(j)$ and hence u_η, v_η have constant values on each J_i. Thus

$$\begin{aligned}\gamma_p \int_I |u_\eta||v_\eta|dt &= \gamma_p \int_{I_1} |u_\eta||v_\eta|dt + \gamma_p \int_{I\setminus I_1} |u_\eta||v_\eta|dt \\ &= \sum_{J_i \subseteq I_1} \|T_{a,J_i,u_\eta,u_\eta}\|_{p,J_i} + \sum_{J_i \not\subseteq I_1} \mathscr{A}(J_i; u_\eta, v_\eta).\end{aligned}$$

We again use Lemmas 6.8–6.10 to obtain

$$\begin{aligned}\sum_{J_i \not\subseteq I_1} \mathscr{A}(J_i; u_\eta, v_\eta) &\leq \sum_{J_i \not\subseteq I_1} \Big\{ \mathscr{A}(J_i; u, v) + [\mathscr{A}(J_i; u, v) - \mathscr{A}(J_i; u_\eta, v)] \\ &\qquad + [\mathscr{A}(J_i; u_\eta, v_\eta) - \mathscr{A}(J_i; u_\eta, v)] \Big\} \\ &\leq \sum_{J_i \not\subseteq I_1} \Big\{ \mathscr{A}(J_i; u, v) + \|u - u_\eta\|_{p',J_i} \|v\|_{p,J_i} \\ &\qquad + \|u_\eta\|_{p',I} \|v_\eta - v\|_{p,I} \Big\},\end{aligned}$$

and with the help of Remark 6.1 and by obvious modification for $\|T_{a,I,u_\eta,v_\eta}\|_{p,I}$ we have

$$\begin{aligned}\sum_{J_i \subseteq I_1} \|T_{a,J_i,u_\eta,v_\eta}\|_{p,J_i} &\leq \sum_{J_i \subseteq I_1} \Big\{ \|T_{a,J_i,u,v}\|_{p,J_i} \\ &\qquad + \left[\|T_{a,J_i,u,v}\|_{p,J_i} - \|T_{a,J_i,u_\eta,v}\|_{p,J_i}\right] \\ &\qquad + \left[\|T_{a,J_i,u_\eta,v_\eta}\|_{p,J_i} - \|T_{a,J_i,u_\eta,v}\|_{p,J_i}\right] \Big\} \\ &\leq \sum_{J_i \subseteq I_1} \Big\{ \|T_{a,J_i,u,v}\|_{p,J_i} + \|u - u_\eta\|_{p',J_i} \|v\|_{p,J_i} \\ &\qquad + \|u_\eta\|_{p',I} \|v_\eta - v\|_{p,I} \Big\}.\end{aligned}$$

Hence, from $\|T_{a,J_i,u,v}\|_{p,J_i} \leq \varepsilon$ and $\mathscr{A}(J_i; u, v) \leq \varepsilon$,

$$\gamma_p \int_I |u(t)||v(t)|dt \leq (N+1+m)\varepsilon + 3\eta \|v\|_{p,I} + \eta(3\|u\|_{p',I} + \eta)$$

and since $\eta > 0$ is arbitrary the theorem follows. □

As an obvious consequence of Lemma 6.14 and Theorems 6.2 and 6.3 we have

Theorem 6.4. *Let* $1 < p < \infty$, $v \in L_p(a,b)$ *and* $u \in L_{p'}(a,b)$. *Then*

$$\gamma_p \int_a^b |u(t)v(t)|dt = \lim_{N\to\infty} \widetilde{s}_N(T)N$$

where γ_p *is as in Theorem 5.8 and* $\widetilde{s}_n$ *stands for any strict s-number.*

6.5 The Cases $p = \infty$ and $p = 1$

In this section we obtain upper and lower estimates for the first asymptotic of the operator $T : L_p \to L_p$ when p is 1 and ∞. The main difficulty in these cases is the absence of strict monotonicity and continuity of A.

Lemma 6.15. *Suppose that (6.2) is satisfied. Then the function* $\mathscr{J}(.,d)$ *given by (6.4) is continuous and non-increasing on* (a,d), *for any* $d < b$ *when* $p = \infty$; *and* $\mathscr{J}(e,.)$ *is continuous and non-increasing on* (e,b), *for any* $e > a$ *when* $p = 1$.

Proof. We give the proof only when $p = \infty$ as that when $p = 1$ follows by a simple modification. Given $x \in (a,b)$ and $\varepsilon > 0$, there exists $h = h(x,\varepsilon) \in \left(0, \min\left\{\frac{1}{2}(x+a), b-x\right\}\right)$ such that

$$\int_{x-h}^{x+h} |u(t)|dt < \min\left(\frac{\varepsilon}{\|v\|_{\infty,(\frac{x-a}{2},d)} + 1}, \varepsilon\right).$$

Then

$$\begin{aligned}
\mathscr{J}(x,d) \le \mathscr{J}(x-h,d) &= \max\left\{\sup_{x-h<z<x}\left[\|v\|_{\infty,(z,d)}\int_{x-h}^{z}|u(t)|dt\right],\right.\\
&\left.\sup_{x<z<d}\left[\|v\|_{\infty,(z,d)}\left(\int_{x-h}^{x}+\int_{x}^{z}\right)|u(t)|dt\right]\right\}\\
&\le \max\{\varepsilon, \varepsilon + \mathscr{J}(x,d)\} = \varepsilon + \mathscr{J}(x,d) \qquad (6.15)
\end{aligned}$$

and so $0 < \mathscr{J}(x-h,a) - \mathscr{J}(x,d) < \varepsilon$. Similarly $0 < \mathscr{J}(x) - \mathscr{J}(x+h) < \varepsilon$ and the continuity is established. It is obvious that $\mathscr{J}(.,d)$ is non-increasing and hence the lemma is proved. □

First we deal with the case $p = \infty$. We now define, for any interval $I \subseteq (a,b)$ and $\varepsilon > 0$,

$$M(I,\varepsilon) := \inf\{n : I = \cup_{i=1}^{n} I_i, \mathscr{A}(I_i) \le \varepsilon\}. \qquad (6.16)$$

Lemma 6.16. *Suppose that (6.2) is satisfied and let* $M(I,\varepsilon) = m < \infty$ *for* $I \subseteq (a,b)$ *and* $\varepsilon > 0$. *Then:*

(i) If $m = 2n$, there exist intervals $J_i, i = 1,2,\ldots,n$, such that $I = \cup_{i=1}^{n} J_i$ and $\mathscr{A}(J_i) > \varepsilon$.
(ii) If $m = 2n+1$, there exist intervals $J_i, i = 1,2,\ldots,n+1$ such that $I = \cup_{i=1}^{n+1} J_i$, $\mathscr{A}(J_i) > \varepsilon, i = 1,\ldots,n$ and $\mathscr{A}(J_{n+1}) \le \varepsilon$.

Proof. From the definition of $M(I,\varepsilon)$ in (6.16) there exist I_i, $i = 1,2,\ldots,m$, such that $\mathscr{A}(I_i) \le \varepsilon$ and $\mathscr{A}(I_i \cup I_{i+1}) > \varepsilon$. Now set $J_1 = I_1 \cup I_2, J_2 = I_3 \cup I_4, \ldots,$ with $J_{n+1} = I_m$ in case (ii). □

The next lemma will yield a one-dimensional approximation to T on I.

Lemma 6.17. *There exists $\omega_I \in \{L_\infty(I)\}^*$ such that $\omega_I(1) = 1$, $\|\omega_I\|_{\{L_\infty(I)\}^*} = 1$ and, for all $f \in L_\infty(I)$,*

$$\inf_{\alpha\in\mathbb{R}} \|(f-\alpha)v\|_{\infty,I} \le \|(f-\omega_I(f))v\|_{\infty,I} \le 2 \inf_{\alpha\in\mathbb{R}} \|(f-\alpha)v\|_{\infty,I}. \tag{6.17}$$

Proof. For $0 < \gamma < \|v\|_{\infty,I}$ and $A_\gamma := \{x : v(x) > \gamma\}$, define $\omega_\gamma \in \{L_\infty(I)\}^*$ by

$$\omega_\gamma(f) := \frac{1}{|A_\gamma|}\int_{A_\gamma} f(x)dx,\ f \in L_\infty(I).$$

Then $\omega_\gamma(1) = 1, \|\omega_\gamma\|_{\{L_\infty(I)\}^*} = 1$ and

$$\left|\omega_\gamma(f)\right| \le \frac{1}{\gamma}\|fv\|_{\infty,I}. \tag{6.18}$$

The set $W := \left\{W_\beta : 0 < \beta < \|v\|_{\infty,I}\right\}$, where $W_\beta = \left\{\omega_\gamma : \gamma > \beta\right\}$, is a filter base whose members W_β are subsets of the unit ball in $\{L_\infty(I)\}^*$. Hence, by the weak* compactness of this unit ball, W has an adherent point, ω_I say. It follows that $\omega_I(1) = 1, \|\omega_I\|_{\{L_\infty(I)\}^*} = 1$ and, from (6.18), for all $\beta \in (0, \|v\|_{\infty,I})$,

$$|\omega_I(f)| \le \frac{1}{\beta}\|fv\|_{\infty,I}, \qquad f \in L_\infty(I).$$

Consequently, for any $\delta \in \mathbb{R}$,

$$\begin{aligned}\inf_{\alpha\in\mathbb{R}} \|(f-\alpha)v\|_{\infty,I} &\le \|(f-\omega_I(f))v\|_{\infty,I}\\ &\le \|(f-\delta)v\|_{\infty,I} + \|\omega_I(f-\delta)v\|_{\infty,I}\\ &\le \|(f-\delta)v\|_{\infty,I}\left\{1 + \frac{\|v\|_{\infty,I}}{\beta}\right\}.\end{aligned}$$

Since $\delta \in \mathbb{R}$ and $\beta \in (0, \|v\|_{\infty,I})$ are arbitrary, the lemma follows. □

The next two lemmas give lower and upper estimates for strict s-numbers which are analogues of those obtained in the case $1 < p < \infty$. Hereafter, we shall always assume (6.2).

Lemma 6.18. *Suppose that $T : L_\infty(a,b) \to L_\infty(a,b)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbb{N}$ and numbers $c_k, k = 0,1,\dots,N$, with $a = c_0 < c_1 < \dots < c_N = b$, such that $\mathscr{A}(I_k) \le \varepsilon$ for $k = 0,1,\dots,N-1$, where $I_k = (c_k, c_{k+1})$. Then $a_{N+1}(T) \le 2\varepsilon$.*

Proof. Let $f \in L_\infty(a,b)$ be such that $\|f\|_\infty = 1$, and write

$$Pf := \sum_{i=0}^{N-1} P_{I_k} f$$

where the P_{I_k} are the one-dimensional operators

$$P_{I_k} f(x) := \chi_{I_k}(x) v(x) \hat{\omega}_{I_k}\left(\int_a^x uf dt\right), \quad k = 0,1,\dots N-1,$$

where

$$\hat{\omega}_{I_k}\left(\int_a^x uf dt\right) = \int_a^{c_k} uf dt + \omega_{I_k}\left(\int_{c_k}^x uf dt\right).$$

with $\omega_{I_k} \in \{L_\infty(I_k)\}^*$ the functionals in Lemma 6.17.

It is obvious that $P_k, k = 1,\dots,N-2$, are bounded. With $k = 0$ or $N-1$ we have on $I = (a, c_1)$ or (c_N, b),

$$\left| v(x)\omega_I\left(\int_{c_k}^x uf dt\right)\right| \le \|\omega_I\|_{\{L_\infty(I)\}^*} \, |v(x)| \int_{c_k}^x |u(t)| dt \, \|f\|_{\infty,I}$$

and hence P is bounded in view of (6.3) and (6.6). We have

$$\begin{aligned}
\|Tf - Pf\|_\infty &= \sup_{k \in \{0,1,\dots,N-1\}} \|Tf - P_{I_k} f\|_{\infty,I_k} \\
&= \sup_{k \in \{0,1,\dots,N-1\}} \|v(x)\left[\int_{c_k}^x uf dt - \omega_{I_k}\left(\int_{c_k}^x uf dt\right)\right]\|_{\infty,I_k} \\
&\le 2 \sup_{k \in \{0,1,\dots,N-1\}} \mathscr{A}(I_k)\|f\|_{\infty,I_k} \le 2\varepsilon \|f\|_{\infty,I}.
\end{aligned}$$

by Lemma 6.17. Since rank$P \le N$, the lemma follows. □

Lemma 6.19. *Suppose that $T : L_\infty(a,b) \to L_\infty(a,b)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbb{N}$ and numbers $d_k, k = 0,1,\dots,K$, with $a = c_0 < c_1 < \dots < c_K \le b$ such that $\mathscr{A}(I_k) \ge \varepsilon$ for $k = 0,1,\dots,K-1$, where $I_k = (c_k, c_{k+1})$. Then $a_K(T) \ge \varepsilon$.*

Proof. Let $\lambda \in (0,1)$. From the definition of $\mathscr{A}(I_k)$ we see that there exists $\phi_k \in L_\infty(I_k)$ with $\|\phi_k\|_{\infty,I_k} = 1$ and such that

$$\inf_{\alpha \in \mathbb{R}} \|T\phi_k - \alpha v\,\|_{\infty,I_k} > \lambda \mathscr{A}(I_k) \ge \lambda\varepsilon. \tag{6.19}$$

Set $\phi_k(x) = 0$ for $x \notin I_k$. Let $P : L_\infty(a,b) \to L_\infty(a,b)$ be bounded and rank $P \le K-1$. Then there are constants $\lambda_0, \ldots, \lambda_{n-1}$, not all zero, such that

$$P\left(\sum_{k=0}^{K-1} \lambda_k \phi_k\right) = 0.$$

Put $\phi = \sum_{k=0}^{K-1} \lambda_k \phi_k$. Then

$$\begin{aligned}
\|T\phi - P\phi\|_\infty &= \|T\phi\|_\infty \\
&\ge \sup_{k\in\{0,1,\ldots,K-1\}} \|v(x)\left(\int_{c_k}^{x} \lambda_k \phi_k(t)u(t)dt + \int_a^{c_k} \phi(t)u(t)dt\right)\|_{\infty,I_k} \\
&= \sup_{k\in\{0,1,\ldots,K-1\}} |\lambda_k| \|T\phi_k + \alpha_k v\|_{\infty,I_k} \\
&\quad (\text{ where } \alpha_k = \lambda_k^{-1} \int_a^{c_k} \phi(t)u(t)dt\ ) \\
&\ge \sup_{k\in\{0,1,\ldots,K-1\}} \inf_{\alpha\in\mathbb{R}} |\lambda_k| \|T\phi_k - \alpha v\|_{\infty,I_k} \\
&\ge \sup_{k\in\{0,1,\ldots,K-1\}} \lambda |\lambda_k| \varepsilon \\
&= \lambda\varepsilon \|\phi\|_\infty
\end{aligned}$$

by (3.1). This implies that $a_K(T) \ge \lambda\varepsilon$, whence the result since $\lambda \in (0,1)$ is arbitrary. □

Corollary 6.2. *Suppose that T is compact. Then, for $\varepsilon \in (0, \mathscr{A}(a,b))$,*

$$a_{M(\varepsilon)+1}(T) \le 2\varepsilon,$$
$$a_{[\frac{M(\varepsilon)}{2}]-1}(T) > \varepsilon,$$

where $M_\varepsilon \equiv M((a,b),\varepsilon)$ is defined in (6.16) and $[.]$ denotes integer part.

Proof. This is an immediate consequence of Lemmas 6.18 and 6.19. □

To continue, we need some preliminary results and the functions v_s given by

$$v_s(x) := \lim_{\varepsilon\to 0_+} \|v\|_{\infty,(x-\varepsilon,x+\varepsilon)} \tag{6.20}$$

for $x \in (a,b)$.

Lemma 6.20. *For any interval $I \subseteq (a,b)$, $\mathscr{J}(I;u,v) = \mathscr{J}(I;u,v_s)$ and $\mathscr{A}(I;u,v) = \mathscr{A}(I;u,v_s)$, where $\mathscr{J}(I;u,v)$ and $\mathscr{A}(I;u,v)$ are the functions defined in (6.4) and Definition 6.1 respectively.*

Proof. For any continuous function ϕ, it is readily shown that $\|v_s\phi\|_{\infty,I} = \|v\phi\|_{\infty,I}$, and this fact yields the lemma. □

Lemma 6.21. *Let $\bar{I} \subset (a,b)$, and let $\vartheta_n = \{I_i^n\}_{i=1}^{l(n)}$ be a partition of I by intervals I_i^n which are such that each $I_i^{(n+1)} \in \vartheta_{n+1}$ is a subinterval of some $I_j^{(n)} \in \vartheta_n$, and $|I_i^n| \to 0$ as $n \to \infty$. Define*

$$v_s^n(t) := \sum_{i=1}^{l(n)} \chi_{I_i^n}(t) c_i^n, \quad c_i^n = \|v_s\|_{\infty, I_i^n}.$$

Then for a.e. $t \in I$,

(i) $\|v_s\|_{\infty,I} \geq v_s^n(t) \geq v_s(t)$,
(ii) $v_s^n(t) \searrow v_s(t)$ *as* $n \to \infty$,
(iii) $\lim_{n\to\infty} \int_I u(t)[v_s^n(t) - v_s(t)]dt = 0$.

Proof. Since v_s is upper semi-continuous and bounded, it is known that it can be approximated from above by a decreasing sequence of step functions. However, we shall give a proof of the lemma for completeness and subsequent reference.

If $t \in \operatorname{int} I_i^n$, the interior of I_i^n, then $v_s^n(t) = \|v_s\|_{\infty, I_i^n}$ satisfies

$$v_s(t) \leq v_s^n(t) \leq \|v_s\|_{\infty,I}.$$

This establishes (i), the exceptional set being $S = \cup_{n\in\mathbb{N}} S_n$, where S_n is the set of end points of the intervals $I_i^n \in \vartheta_n$. If $t \in \operatorname{int} I_{i(n+1)}^{n+1} \subset \operatorname{int} I_{i(n)}^n$ say, we have $c_{i(n+1)}^{n+1} \leq c_{i(n)}^n$ and so $v_s^{n+1}(t) \leq v_s^n(t)$ for $t \in I \setminus S$. Also, if $t \in \operatorname{int} I_{i(n)}^n$,

$$v_s^n(t) = \|v_s\|_{\infty, I_{i(n)}^n} = \|v\|_{\infty, I_{i(n)}^n} \geq v(t)$$

as observed in the proof of Lemma 6.20. Moreover, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$v_s(t) > \|v\|_{\infty,(t-\varepsilon_0, t+\varepsilon_0)} - \delta.$$

Now choose N such that for all $n \geq N$,

$$t \in \operatorname{int} I_{i(n)}^n \subset (t - \varepsilon_0, t + \varepsilon_0).$$

Then we have that for all $n \geq N$,

$$0 < v_s^n(t) - v_s(t) < \delta$$

and hence $v_s^n(t) \to v_s(t)$ for all $t \in I \setminus S$.

Finally, (iii) follows by the dominated convergence theorem since $u \in L_1(I)$ and $\|v_s^n\|_{\infty,I} = \|v_s\|_{\infty,I} = \|v\|_{\infty,I} < \infty$. □

Lemma 6.22. *Let u, v be constant on $\bar{I} \subset (a,b)$. Then*

$$\mathscr{A}(I) = \frac{1}{2}|u||v||I|. \tag{6.21}$$

Proof. We have if $I = (c,d)$,

$$\begin{aligned}\mathscr{A}(I) &\geq |u||v| \inf_{\alpha} \|x - c - \alpha\|_{\infty,I} \\ &= |u||v| \|x - c - \frac{1}{2}(d-c)\|_{\infty,I} \\ &= \frac{1}{2}|u||v||I|.\end{aligned}$$

Let $f \in L_\infty(I)$ and set $F(x) = \int_c^x f\,dt$. Then there exist $x_0, x_1 \in [c,d]$ such that

$$F(x_0) \leq F(x) \leq F(x_1), \qquad x \in [a,b]$$

and hence

$$\begin{aligned}\inf_{\alpha} \|F - \alpha\|_{\infty,I} &\leq \|F - \frac{1}{2}\left(F(x_0) + F(x_1)\right)\|_{\infty,I} \\ &= \frac{1}{2}\left(F(x_1) - F(x_0)\right) \\ &= \frac{1}{2}\int_{x_0}^{x_1} f\,dt.\end{aligned}$$

This yields

$$\mathscr{A}(I) \leq |u||v| \sup_{\|f\|_{\infty,I}=1} \left\{ \frac{1}{2}\int_{x_0}^{x_1} f\,dt \right\} \leq \frac{1}{2}|u||v||I|$$

and the lemma is proved. □

In the next lemma g^* denotes the non-increasing rearrangement of a function g on an interval I: g^* is the generalised inverse of the non-increasing distribution function g_* of g, namely

$$g^*(x) := \inf\{t : g_*(t) \geq x\} \tag{6.22}$$

where

$$g_*(t) := |\{x \in I : g(x) \geq t\}|. \tag{6.23}$$

Note that since we have $\geq$ in the definitions above, g_* and g^* are left-continuous functions.

Lemma 6.23. *Let $\bar{I} \subset (a,b)$ and $\gamma, \delta \in \mathbb{R}$ with $\delta \geq v_s(t) \geq 0$ on I. Then*

$$\mathscr{A}(I;\gamma,\delta) \geq \mathscr{A}(I;\gamma,v_s) \geq \frac{1}{2}|\gamma| \|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}. \tag{6.24}$$

Proof. The first inequality in (6.24) is obvious. The set

$$M_\beta := \{y \in I : v_s(y) \geq \beta\}$$

is relatively closed in $\bar{I}$. For if $\{y_n\} \subset M_\beta$ and $y_n \to y \in \bar{I}$ as $n \to \infty$, then given $\varepsilon > 0$ there exists N such that $(y-\varepsilon, y+\varepsilon) \supset (y_n - \frac{1}{2}\varepsilon, y_n + \frac{1}{2}\varepsilon)$ for $n > N$. Hence

$$\|v\|_{\infty,(y-\varepsilon,y+\varepsilon)} \geq \|v\|_{\infty,(y_n-\frac{1}{2}\varepsilon,y_n+\frac{1}{2}\varepsilon)} \geq v_s(y_n) \geq \beta,$$

whence $v_s(y) \geq \beta$ and $y \in M_\beta$. From the observed left continuity ensured by (6.22) and (6.23), we have

$$\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)} = \max_{(0,|I|]} |(v_s\chi_I)^*(t)t| = |(v_s\chi_I)^*(t_0)t_0|$$

for some $t_0 \in (0,|I|]$, and there exist $\beta > 0$ such that $|M_\beta| = t_0$. Choose the optimal c_0, d_0 such that $M_\beta \subseteq [c_0, d_0] \subseteq \bar{I}$. Then, with $I = (c,d)$,

$$\begin{aligned}
\mathscr{A}(I;\gamma,v_s) &\geq |\gamma| \inf_\alpha \|v_s(y)\left(\int_c^y dt - \alpha\right)\|_{\infty,I} \\
&\geq |\gamma| \inf_\alpha \|\beta\chi_{M_\beta}(y)(y-c-\alpha)\|_{\infty,I} \\
&= \beta|\gamma|\|y-c-\frac{1}{2}(c_0+d_0-2c)\|_{\infty,M_\beta} \\
&= \frac{1}{2}\beta|\gamma|(d_0-c_0) \;\geq\; \frac{1}{2}\beta|\gamma||M_\beta| \\
&= \frac{1}{2}|\gamma|\,[(v_s\chi_I)^*(t_0)t_0] \;=\; \frac{1}{2}|\gamma|\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)}.
\end{aligned}$$

The lemma is therefore proved. □

Lemma 6.24. *Let $\bar{I} \subset (a,b)$ and $\gamma, \delta \in \mathbb{R}$ with $\delta \geq v_s(t) \geq 0$ on I. Then, for any $\alpha > 1$,*

$$\mathscr{A}(I;\gamma,\delta) - \mathscr{A}(I;\gamma,v_s) \leq \frac{\alpha}{2}\int_I |\gamma|\,(\delta - v_s(t))\,dt + \frac{|\gamma|\delta|I|}{2\alpha}. \tag{6.25}$$

Proof. We first observe that

$$(v_s\chi_I)^*(t) \geq v_0(t) := \left(\delta - \frac{V\alpha}{|\gamma||I|}\right)\chi_{(0,|I|-\frac{|I|}{\alpha})}(t) \tag{6.26}$$

where $V = |\gamma|\int_I(\delta - v_s(t))dt$. For, with $S := \left\{x : v_s(x) < \delta - \frac{V\alpha}{|\gamma||I|}\right\}$,

$$\frac{V}{|\gamma|} > \int_S \left(\delta - \delta + \frac{V\alpha}{|\gamma||I|}\right)dt = \frac{V\alpha}{|\gamma||I|}|S|,$$

which implies that

$$\left|\left\{x : v_s(x) > \delta - \frac{V\alpha}{|\gamma||I|}\right\}\right| > |I| - \frac{|I|}{\alpha}$$

and hence (6.26). Note that (6.26) is trivially true if $\delta - \frac{V\alpha}{|\gamma||I|} < 0$. On using (6.21) and (6.24),

$$
\begin{aligned}
0 \le \mathscr{A}(I;\gamma,\delta) - \mathscr{A}(I;\gamma,v_s) &\le \frac{1}{2}|\gamma|\delta|I| - \frac{1}{2}|\gamma|\|(v_s\chi_I)^*(t)t\|_{\infty,(0,|I|)} \\
&\le \frac{1}{2}|\gamma|\delta|I| - \frac{1}{2}\max_{(0,|I|]}(tv_0(t)) \\
&= \frac{1}{2}|\gamma|\delta|I| - \frac{1}{2}|\gamma|\left(\delta - \frac{V\alpha}{|\gamma||I|}\right)\left(|I| - \frac{|I|}{\alpha}\right) \\
&= \frac{\alpha V}{2} + \frac{|\gamma|\delta|I|}{2\alpha} - \frac{V}{2} \\
&\le \frac{\alpha}{2}\int_I |\gamma|\left(\delta - v_s(t)\right)dt + \frac{|I|}{2\alpha}|\gamma|\delta
\end{aligned}
$$

which is (6.25). □

Theorem 6.5. *Suppose $u \in L_1(I)$ and $v \in L_\infty(I)$. Then*

$$
\frac{1}{2}\int_I |u(t)|v_s(t)dt \le \liminf_{\varepsilon\to 0_+} \varepsilon M(I,\varepsilon) \le \limsup_{\varepsilon\to 0_+} \varepsilon M(I,\varepsilon) \le \int_I |u(t)|v_s(t)dt. \quad (6.27)
$$

Proof. On using Lemma 6.21, we see that for each $\eta > 0$ there exist step functions u_η, v_η on I such that

$$
\begin{aligned}
\|u - u_\eta\|_{1,I} &< \eta, \\
\int_I |u(t)|\left(v_\eta(t) - v_s(t)\right)dt &< \eta
\end{aligned}
$$

and

$$
\|v_s\|_{\infty,I} \ge v_\eta(t) \ge v_s(t)
$$

on I. We may assume that

$$
u_\eta = \sum_{j=1}^m \xi_j \chi_{W(j)}, \qquad v_\eta = \sum_{j=1}^m \eta_j \chi_{W(j)},
$$

where the $W(j)$ are disjoint subintervals of I, and $\eta_j \ge 0$.

Let $\varepsilon > 0, M \equiv M(I,\varepsilon)$, and let $c_k \equiv c_k(\varepsilon), k = 1,2,\ldots,M+1$, be the end-points of the intervals in (6.16): with $I = [c,d]$ and $I_k \equiv I_k(\varepsilon) = [c_k, c_{k+1}]$, we have $c = c_1 < c_2 < \cdots < c_{M+1} = d$ and

$$
\begin{aligned}
\mathscr{A}(I_k) \equiv \mathscr{A}(I_k;u,v) &\le \varepsilon \quad, k = 1,2,\ldots,M, \\
\mathscr{A}(I_k \cup I_{k+1}) &> \varepsilon \quad, k = 1,2,\ldots,M-1.
\end{aligned}
$$

Then

$$\begin{aligned}\left|\int_I |u(t)|\, v_s(t)dt - \int_I |u_\eta(t)| v_\eta(t)dt\right| &\\ \le \int_I |u(t)|\,(v_\eta(t)-v_s(t))\,dt + \int_I |u(t)-u_\eta(t)| v_\eta(t)dt &\\ < \eta(1+\|v_\eta\|_{\infty,I}) &\\ \le \eta(1+\|v_s\|_{\infty,I}). &\end{aligned} \tag{6.28}$$

Next, let $\mathbb{K} := \{k\colon \text{there exists } j \text{ such that } I_{2k}\cup I_{2k+1}\subset W(j)\}$. Then $\#\mathbb{K} \ge [\frac{M}{2}]-2m \ge \frac{M}{2}-1-2m$, and, by Lemmas 6.9 and 6.24,

$$\begin{aligned}\left(\frac{M}{2}-1-2m\right)\varepsilon &\le \sum_{k\in\mathbb{K}} \mathscr{A}(I_{2k}\cup I_{2k+1};u,v)\\ &\le \sum_{k\in\mathbb{K}} \big\{\mathscr{A}(I_{2k}\cup I_{2k+1};u_\eta,v_\eta)\\ &\qquad + (\mathscr{A}(I_{2k}\cup I_{2k+1};u,v_s) - \mathscr{A}(I_{2k}\cup I_{2k+1};u_\eta,v_s))\\ &\qquad + (\mathscr{A}(I_{2k}\cup I_{2k+1};u_\eta,v_s) - \mathscr{A}(I_{2k}\cup I_{2k+1};u_\eta,v_\eta))\big\}\\ &\le \frac{1}{2}\sum_j |\xi_j|\eta_j|W(j)|\\ &\qquad + \sum_j \big\{\|u-u_\eta\|_{1,W(j)}\|v_s\|_{\infty,W(j)}\\ &\qquad\qquad + \frac{\alpha}{2}\int_{W(j)} |\xi_j|(v_\eta - v_s)dt + \frac{|\xi_j|\;\eta_j}{2\alpha}|W(j)|\big\}\\ &\le \frac{1}{2}\int_I |u_\eta| v_\eta dt + \|u-u_\eta\|_{1,I}\|v_s\|_{\infty,I}\\ &\qquad + \frac{\alpha}{2}\int_I |u_\eta|(v_\eta - v_s)dt + \frac{1}{2\alpha}\int_I |u_\eta| v_\eta dt\\ &\le \frac{1}{2}\int_I |u_\eta| v_\eta dt + K\left(\alpha\eta + \frac{1}{\alpha}\right)\\ &\le \frac{1}{2}\int_I |u(t)| v_s(t)dt + K\left(\alpha\eta + \frac{1}{\alpha}\right)\end{aligned}$$

by (6.28), for some constant K independent of ε. We therefore conclude that

$$\limsup_{\varepsilon\to 0_+} \varepsilon M(I,\varepsilon) \le \int_I |u(t)| v_s(t)dt + K\left(\alpha\eta + \frac{1}{\alpha}\right)$$

and the right-hand inequality in (6.27) follows since $\eta > 0$ and $\alpha > 1$ are arbitrary.

For the left-hand inequality in (6.27), we add the end-points of the intervals $W(j), j = 1,2\ldots,m$ to the $c_k, k = 1,2,\ldots,M-1$, to form the partition $c = e_1 < \cdots < e_n = d$, say, where $n \le M+1+m$. Note that each interval $J_i := [e_i, e_{i+1}]$ is a subinterval of some $W(j)$ and hence u_η, v_η have constant values on each J_i. We again use Lemmas 6.9, 6.22, and 6.24 to get

$$\begin{aligned}\frac{1}{2}\int_I |u_\eta| v_\eta dt &= \sum_{j=1}^{m}\sum_{J_i\subseteq W(j)} \mathscr{A}(J_i; u_\eta, v_\eta)\\ &\leq \sum_{i=1}^{n}\Big\{\mathscr{A}(J_i; u, v_s) + \|u-u_\eta\|_{1,J_i}\|v_s\|_{\infty,J_i}\\ &\qquad + \frac{\alpha}{2}\int_{J_i}|u_\eta|(v_\eta - v_s)dt + \frac{1}{2\alpha}\int_{J_i}|u_\eta| v_\eta dt\Big\}\\ &\leq (M+1+m)\varepsilon + K\left(\alpha\eta + \frac{1}{\alpha}\right).\end{aligned}$$

Hence, from (6.28),

$$\frac{1}{2}\int_I |u(t)| v_s(t) dt \leq (M+1+m)\varepsilon + K\left(\alpha\eta + \frac{1}{\alpha}\right)$$

and the left-hand inequality in (6.27) follows. □

One of the main results of this section is

Theorem 6.6. *Suppose that $u \in L_1(a,b)$ and $v \in L_\infty(a,b)$. Then*

$$\frac{1}{4}\int_a^b |u(t)| v_s(t) dt \leq \liminf_{n\to\infty} n a_n(T) \leq \limsup_{n\to\infty} n a_n(T) \leq 2\int_a^b |u(t)| v_s(t) dt. \quad (6.29)$$

Proof. This is an obvious application of Corollary 6.2 and Theorem 6.5. □

Now we consider the case when $p=1$: the assumptions from the case $p=\infty$ that $u \in L_1(a,x)$ and $v \in L_\infty(x,b)$ are replaced by

$$\begin{aligned} u &\in L_\infty(a,x),\\ v &\in L_1(x,b), \end{aligned} \quad (6.30)$$

for all $x \in (a,b)$. On setting $a=-B$, $b=-A$, $\hat{f}(x) = f(-x)$, and similarly for u, v in (6.1), we see that

$$T\hat{f}(x) = \hat{v}\,(x)\int_x^B \hat{u}\,(t)\hat{f}(t)dt, \quad A \leq x \leq B.$$

But this is the adjoint of the map $S: L_\infty(A,B) \to L_\infty(A,B)$ defined by

$$Sg(x) = \hat{u}\,(x)\int_A^x \hat{v}\,(t) g(t) dt, \quad A \leq x \leq B.$$

Hence, T and S have the same norms and their approximation numbers are equal if one, and hence both, are compact (see [41, Proposition II.2.5]). The results for $T : L_1(a,b) \to L_1(a,b)$ therefore follow from those proved for the $L_\infty(a,b)$ case on interchanging u and v.

Theorem 6.7. *Suppose that $u \in L_\infty(a,b)$ and $v \in L_1(a,b)$. Then*

$$\frac{1}{4}\int_a^b u_s(t)|v(t)|dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le 2\int_a^b u_s(t)|v(t)|dt;$$

We conclude this section with the following remark which demonstrates that v_s in the case $p=\infty$ (u_s when $p=1$) is important.

Remark 6.4. Let M be a dense subset of $(0,1)$ with measure $|M| = \alpha < 1$ and let $u = 1, v = \chi_M$. Then $v_s = 1$, $(v - v_s)_s = 1$ on $(0,1)$ and so

$$\|v\|_{\infty,(x,1)} = \|v_s\|_{\infty,(x,1)} = \|v - v_s\|_{\infty,(x,1)}$$

for any $x \in (0,1)$. Since

$$\|T_{u,v}|L_\infty(0,1) \to L_\infty(0,1)\| = \sup_{0<x<1}\left\{\int_0^x dt\|v\|_{\infty,(x,1)}\right\},$$

where $T_{u,v}$ denotes the operator in (6.1), it follows that

$$\|T_{u,v}\| = \|T_{u,v_s}\| = \|T_{u,v} - T_{u,v-v_s}\|,$$

for the operator norms from $L_\infty(0,1)$ to $L_\infty(0,1)$. Also

$$\int_0^1 |u(t)v(t)|dt = |M| < 1 = \int_0^1 |u(t)|v_s(t)dt.$$

The choice $u = \chi_M$, $v = 1$ gives an analogous example in the $L_1(0,1)$ case.

Notes

Note 6.1. The main result in this section, that for the (weighted) Hardy operator, as a map from $L_p(I)$ to itself, all strict s-numbers coincide, was first proved in [51].

Note 6.2. The asymptotic results given here were first established in [43] when $p=2$; see also [100]. The general case, namely that when $1 < p < \infty$, was first proved in [59], where it appears as a special case of results for trees.

Note 6.3. The cases $p = 1, \infty$ dealt with here were established in [58].

Chapter 7
More Refined Estimates

This chapter focuses on two properties of the strict s-numbers of the Hardy operator $T : L_p(a,b) \to L_p(a,b)$: (a) the asymptotic information given in the last chapter can be sharpened when more information about u and v is available; (b) the strict s-numbers form a sequence in l_q or weak-l_q if and only if u and v satisfy appropriate conditions.

7.1 Remainder Estimates

In this section we improve the asymptotic results of Sect. 6.4 concerning the strict s-numbers of $T : L_p(a,b) \to L_p(a,b)$. To do this we will be forced to introduce conditions on the behavior of the derivatives of u and v.

We start this section with some observations about optimal approximation via step functions. Suppose $u \in L_{p'}(a,b)$, $v \in L_p(a,b)$ and let $\alpha > 0$. We define $m_\alpha \in \mathbb{N}$ by the following requirements:

There exist two step-functions, u_α and v_α, each with m_α steps, say,

$$u_\alpha(x) := \sum_{j=1}^{m_\alpha} \xi_j \chi_{w_\alpha(j)}(x), \qquad v_\alpha(x) := \sum_{j=1}^{m_\alpha} \psi_j \chi_{w_\alpha(j)}(x), \tag{7.1}$$

where $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a family of non-overlapping intervals covering (a,b), such that for

$$\alpha_u := \|u - u_\alpha\|_{p',(a,b)} \qquad \text{and} \qquad \alpha_v := \|v - v_\alpha\|_{p,(a,b)}$$

we have

(1)

$$\max(\alpha_u, \alpha_v) \le \alpha; \tag{7.2}$$

and

(2) for any step-functions u'_α, v'_α with less than m_α steps, say n_α steps, $n_\alpha < m_\alpha$,

$$\max(\|u - u'_\alpha\|_{p',(a,b)}, \|v - v'_\alpha\|_{p,(a,b)}) > \alpha.$$

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Thus, m_α is the minimum number of steps needed to approximate u in $L_{p'}$ and v in L_p with the required accuracy. Note that, plainly,

$$\|u-u_\alpha\|_{p',(a,b)} \leq \alpha, \qquad \|v-v_\alpha\|_{p,(a,b)} \leq \alpha.$$

The best way to choose ξ_i and ψ_i for given $\{w_\alpha\}_{j=1}^{m_\alpha}$ is by finding ξ_i and ψ_i such that:

$$\int_{w_\alpha(i)} |u(t)-\xi_i|^{p'-1}\operatorname{sgn}(u(t)-\xi_i)dt = 0$$

and

$$\int_{w_\alpha(i)} |v(t)-\psi_i|^{p-1}\operatorname{sgn}(v(t)-\psi_i)dt = 0$$

(see [115], Theorem 1.11).

It turns out that the relationship between α and m_α is crucial for us; we next address this matter.

Lemma 7.1. *Suppose $u \in C(a,b)\cap L_{p'}(a,b)$ and $v \in C(a,b)\cap L_p(a,b)$, at least one of them, say u, being non-constant. Then, when α decreases to 0, m_α increases to ∞.*

Proof. We show that given $m \in N$ there exists $\alpha > 0$ having $m_\alpha > m$. The fact that u is continuous and non-constant on (a,b) guarantees the existence of pairwise disjoint subintervals $I_1, I_2, \dots, I_{2m}$ of (a,b) on each of which u is non-constant.

Fix $\alpha > 0$ satisfying $\sum_{j=1}^m \|u-u_{I_{k_j}}\|_{p',I_{k_j}}^{p'} > \alpha^{p'}$ for every set of m intervals from among $I_1, I_2, \dots, I_{2m}$. Now, to any partition, $\{w_\alpha(j)\}_{j=1}^m$, of (a,b) into m non-overlapping subintervals there correspond $I_{k_1}, I_{k_2}, \dots, I_{k_m}$ such that each I_{k_j} is a subset of some $w_\alpha(i)$ and hence

$$\sum_{j=1}^m \|u-u_{w_\alpha(j)}\|_{p',w_\alpha(j)}^{p'} \geq \sum_{j=1}^m \|u-u_{w_\alpha(j)}\|_{p',I_{k_j}}^{p'} > \alpha^{p'}.$$

Therefore $m_\alpha > m$. □

Lemma 7.2. *Suppose $u \in C(a,b)\cap L_{p'}(a,b)$ and $v \in C(a,b)\cap L_p(a,b)$, at least one of them, say u, being non-constant. Fix $\alpha > 0$ and set $\Lambda_\alpha = \{\beta : 0 < \beta \leq \alpha$ and $m_\beta = m_\alpha\}$. Then Λ_α is an interval with $\gamma = \inf \Lambda_\alpha$ and $\gamma \in \Lambda_\alpha$.*

Proof. Clearly, Λ_α is nonempty, since $\alpha \in \Lambda_\alpha$. Again, $m_{\lambda_1} \geq m_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$, so Λ_α is convex and hence an interval, possibly equal to $\{\alpha\}$.

It follows from Lemma 7.1 that $\gamma > 0$. Now, if $\Lambda_\lambda = \{\alpha\}$, so that $\gamma = \alpha$, we are done. Otherwise, there exists a sequence $\{\alpha_n\}$ in Λ_α with $\alpha_n \searrow \gamma$. Let $u_{\alpha_n} = \sum_{j=1}^{m_\alpha} u_{w_{\alpha_n}(j)}\chi_{w_{\alpha_n}(j)}$ and $v_{\alpha_n} = \sum_{j=1}^{m_\alpha} v_{w_{\alpha_n}(j)}\chi_{w_{\alpha_n}(j)}$, as in (7.1), so that

$$\max(\|u-u_{\alpha_n}\|_{p',(a,b)}, \|v-v_{\alpha_n}\|_{p,(a,b)}) \leq \alpha_n.$$

Assume the notation has been chosen to ensure the end points of $w_{\alpha_n}(j) = (c_n^j, d_n^j)$ satisfy $a = c_n^1 < d_n^j \leq c_n^{j+1} < d_n^{m_\alpha} = b$, $j = 1,2,\dots,m_\alpha - 1$.

There exists a sequence $n(k), k = 1, 2, \ldots,$ of positive integers and numbers $c^1, c^2, \ldots, c^{m_\alpha}, d^1, d^2, \ldots, d^{m_\alpha}$ such that

$$\lim_k c^j_{n(k)} = c^j, \qquad \lim_k d^j_{n(k)} = d^j, \qquad j = 1, 2, \ldots, m_\alpha,$$

and

$$a = c^1 \le d^j \le c^{j+1} \le d^{m_\alpha} = b, \qquad j = 1, 2, \ldots, m_\alpha.$$

Observe that, setting

$$u_\gamma = \sum_{j=1}^{m_\alpha} u_{(c^j,d^j)} \chi_{(c^j,d^j)} \qquad \text{and} \qquad v_\gamma = \sum_{j=1}^{m_\alpha} v_{(c^j,d^j)} \chi_{(c^j,d^j)},$$

we have

$$\max(\|u - u_\gamma\|_{p',(a,b)}, \|v - v_\gamma\|_{p,(a,b)}) = \gamma,$$

which forces $m_\gamma = m_\alpha$. □

Lemma 7.3. *Suppose that $u \in L_{p'}(a,b) \cap C(a,b)$ and $v \in L_p(a,b) \cap C(a,b)$ are not equal to zero on (a,b); let at least one of u and v be non-constant on (a,b). Then there exists $\alpha_0 > 0$ such that given any α, $0 < \alpha < \alpha_0$, there exists a β, $0 < \beta < \alpha$, with $m_\beta = m_\alpha + 1$ or $m_\beta = m_\alpha + 2$.*

Proof. Say u is non-constant on (a,b). We take α_0 to be the positive distance of u from the closed set $\{k\chi_I; k \in \mathbb{R}, 0 < |I| < \infty\}$ in $L^{p'}(a,b)$. Observe that $m_\alpha \ge 2$ whenever $0 < \alpha < \alpha_0$.

Fix $\alpha, 0 < \alpha < \alpha_0$. By Lemma 7.2, $m_\gamma = m_\alpha$, where $\gamma = \inf \Lambda_\alpha$. Hence, there exists a partition $\{w_\gamma(j)\}_{j=1}^{m_\gamma}$ of (a,b) whose corresponding step functions, $u_\gamma = \sum_{j=1}^{m_\alpha} u_{w_\gamma(j)} \chi_{w_\gamma(j)}$ and $v_\gamma = \sum_{j=1}^{m_\alpha} v_{w_\gamma(j)} \chi_{w_\gamma(j)}$, satisfy

$$\max(\|u - u_\gamma\|_{p',(a,b)}, \|v - v_\gamma\|_{p,(a,b)}) = \gamma.$$

If $\|u - u_\gamma\|_{p',(a,b)} > \|v - v_\gamma\|_{p,(a,b)}$ then for some j_0, $1 \le j_0 \le m_\alpha$,

$$\|u - u_{w_\gamma(j_0)}\|^{p'}_{p',w_\gamma(j_0)} > 0.$$

It is possible to find a point c in the interval $w_\gamma(j_0) = (d,e)$ such that

$$\|u - u_{w_\gamma(j_0)}\|^{p'}_{p',w_\gamma(j_0)} > \|u - u_{(d,c)}\|^{p'}_{p',(d,c)} + \|u - u_{(c,e)}\|^{p'}_{p',(c,e)}.$$

Let $w'_\gamma(j) = w_\gamma(j)$, $j = 1, 2, \ldots, j_0 - 1, j_0 + 1, \ldots, m_\alpha$, $w'_\gamma(j_0) = (d,c)$ and $w'_\gamma(m_\alpha + 1) = (c,e)$. Then, $\{w'_\gamma(j)\}_{j=1}^{m_\alpha+1}$ is a partition of (a,b) with associated step functions $u'_\gamma = \sum_{j=1}^{m_\alpha+1} u_{w'_\gamma(j)} \chi_{w'_\gamma(j)}$ and $v'_\gamma = \sum_{j=1}^{m_\alpha+1} v_{w'_\gamma(j)} \chi_{w'_\gamma(j)}$ such that

$$\max(\|u - u'_\gamma\|_{p',(a,b)}, \|v - v'_\gamma\|_{p,(a,b)}) = \beta < \gamma,$$

and so $m_\beta = m_\alpha + 1$.

Similarly, when $\|v-v_\gamma\|_{p,(a,b)} > \|u-u_\gamma\|_{p,(a,b)}$, there is a $\beta \in (0,\alpha)$ with $m_\beta = m_\alpha + 1$.

Suppose, then, $\|v-v_\gamma\|_{p,(a,b)} = \|u-u_\gamma\|_{p',(a,b)} = \gamma > 0$. As before, we can find an interval $w_\gamma(j_0) = (d_0, e_0)$ and a point c_0 such that

$$\|u-u_{w_\gamma(j_0)}\|^{p'}_{p',w_\gamma(j_0)} > \|u-u_{(d_0,c_0)}\|^{p'}_{p',(d_0,c_0)} + \|u-u_{(c_0,e_0)}\|^{p'}_{p',(c_0,e_0)},$$

and an interval $w_\gamma(j_1) = (d_1, c_1)$ and a point c_1 such that

$$\|v-v_{w_\gamma(j_1)}\|^{p}_{p,w_\gamma(j_1)} > \|v-v_{(d_1,c_1)}\|^{p}_{p,(d_1,c_1)} + \|v-v_{(c_1,e_1)}\|^{p}_{p,(c_1,e_1)}.$$

Now, if it is possible to have $j_0 = j_1$ and $c_0 = c_1$ we can get $\beta \in (0,\alpha)$ with $m_\beta = m_\alpha + 1$. Otherwise, we can only conclude there is a $\beta \in (0,\alpha)$ for which m_β is one of $m_\alpha + 1$ and $m_\alpha + 2$. □

Lemma 7.4. *Let $-\infty \le a < b \le \infty$ and suppose that $u' \in L^{p'/(p'+1)}(a,b) \cap C(a,b)$. For each small $h > 0$ define*

$$x_1 = -\frac{1}{h}, x_{i+1} := x_i + h \text{ for } i \in 1,\dots,[2/h^2];$$

put $J_i = (a,b) \cap (x_i, x_{i+1})$, $i \in 1,\dots,[2/h^2]$.

Then

$$\int_a^b |u'(t)|^{p'/(p'+1)} dt = \lim_{h\to 0} \sum_{i=1}^{[2/h^2]} |J_i| \max_{x\in J_i} |u'(x)|^{p'/(p'+1)}$$
$$= \lim_{h\to 0} \sum_{j=1}^{[2/h^2]} |J_i| \min_{x\in J_i} |u'(x)|^{p'/(p'+1)}.$$

Proof. Simply use the definition of the integral. □

We are now prepared to establish an important estimate for $\limsup_{\alpha\to 0_+} \alpha m_\alpha$.

Theorem 7.1. *Suppose $u \in L_{p'}(a,b)$, $v \in L_p(a,b)$ and $u' \in L_{p'/(p'+1)}(a,b) \cap C(a,b)$, $v' \in L_{p/(p+1)}(a,b) \cap C(a,b)$. Then*

$$\limsup_{\alpha\to 0_+} \alpha m_\alpha \le c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)}).$$

Proof. As the result is trivial if both u and v are constant we assume that at least one of them, say u, is not.

Given $\beta, 0 < \beta < \inf_{c\in\mathbb{R}} \|u-c\|_{p',(a,b)}$, let $w_\beta(i) = (a_i, a_{i+1})$, $i = 1,2,\dots,n^u_\beta$, be a partition of (a,b) satisfying

$$\|u-u_{w_\beta(i)}\|_{p',w_\beta(i)} = \beta, \qquad i = 1,2,\dots,n^u_\beta - 1,$$

and $\|u-u_{w_\beta(i)}\|_{p',w_\beta(i)} \le \beta$, $i=n^u_\beta$. Fix λ, $0<\lambda<1$, and define the $[\lambda n^u_\beta]$ points x_k by the rule that if (a,b) is bounded, then

$$x_k := a + \frac{b-a}{\lambda n^u_\beta}k, \qquad k=1,2,\dots,[\lambda n^u_\beta];$$

if $(a,b)=(-\infty,\infty)$, then, with $h=(\frac{2}{\lambda n^u_\beta})^{1/2}$,

$$x_1 = -\frac{1}{h},\ \ x_{k+1}=x_k+h, \qquad k=1,2,\dots,[\lambda n^u_\beta];$$

for other types of intervals we proceed in the same sort of way.

From the union of the points $a_1,a_2,\dots,a_{n^u_\beta}+1$ and $x_1,x_2,\dots,x_{[\lambda n^u_\beta]}$, arrange them in ascending order and denote the resulting points by $b_j, j=1,2,\dots,J(\beta)+1$, so that $n^u_\beta \le J(\beta) \le n^u_\beta + [\lambda n^u_\beta]$. Put $I^\beta_j = (b_j,b_{j+1}), j=1,2,\dots,J(\beta)$. We observe there are at least $n^u_\beta - [\lambda n^u_\beta]$ intervals I^β_j with

$$I^\beta_j = w_\beta(i)$$

for some i.

Now,

$$\sum_{j=1}^{J(\beta)} \|u-u_{I^\beta_j}\|^{p'/(p'+1)}_{p',I^\beta_j} \le \sum_{j=1}^{J(\beta)} |I^\beta_j| \max_{x\in I^\beta_j} |u'(x)|^{p'/(p'+1)}.$$

Again, setting $N=\#\{j: I^\beta_j = w_\beta(i) \ \text{ for some } i<n^u_\beta\}$, we have $N\ge n^u_\beta - [\lambda n^u_\beta]-1$ and

$$\begin{aligned}\beta^{p'/(p'+1)}(n^u_\beta - [\lambda n^u_\beta]-1) \le \beta^{p'/(p'+1)}N &\le \sum_{j=1}^{J(\beta)} \|u-u_{I^\beta_j}\|^{p'/(p'+1)}_{p',I^\beta_j} \\ &\le \sum_{j=1}^{J(\beta)} |I^\beta_j| \max_{x\in I^\beta_j} |u'(x)|^{p'/(p'+1)}.\end{aligned}$$

Thus, by Lemma 7.4,

$$\limsup_{\beta\to 0_+} \beta^{p'/(p'+1)}(n^u_\beta - [\lambda n^u_\beta]) \le \int_a^b |u'(x)|^{p'/(p'+1)}dx. \tag{7.3}$$

Similarly, if neither v is constant, there exists, for $0<\beta<\inf_{c\in\mathbb{R}}\|v-c\|_{p,(a,b)}$, a partition $\{w'_\beta(i)\}_{i=1}^{n^v_\beta}$ such that

$$\|v - v_{w'_\beta(i)}\|_{p,w'_\beta(i)} = \beta, \qquad i = 1,2,\dots,n^v_\beta - 1,$$
$$\|v - v_{w'_\beta(i)}\|_{p,w'_\beta(i)} \le \beta, \qquad i = n^v_\beta,$$

and

$$\limsup_{\beta\to 0_+} \beta^{p/(p+1)}(n^v_\beta - [\lambda n^v_\beta]) \le \int_a^b |v'(x)|^{p/(p+1)} dx. \tag{7.4}$$

Put $\alpha = \max[(\beta^{p'}(n_\beta + [\lambda n_\beta]))^{1/p'}, (\beta^p(n_\beta + [\lambda n_\beta]))^{1/p}]$,

$$0 < \beta < \min[\inf_{c\in\mathbb{R}} \|u - c\|_{p',(a,b)}, \inf_{c\in\mathbb{R}} \|v - c\|_{p,(a,b)}],$$

where $n_\beta = n^u_\beta + n^v_\beta$ if v is not constant and $n_\beta = n^u_\beta$ if it is. Note that (7.3) and (7.4) imply $\alpha \to 0_+$ as $\beta \to 0_+$.

Taking the refinement of the partition $\{I^\beta_j\}_{j=1}^{J(\beta)}$ and the analogous one for v (if necessary) we get a partition of (a,b), of at most $n_\beta + [\lambda n_\beta]$ subintervals, whose corresponding step-functions u_α and v_α satisfy

$$\max[\|u - u_\alpha\|_{p',(a,b)}, \|v - v_\alpha\|_{p,(a,b)}] \le \beta \max[(n^u_\beta)^{1/p'}, (n^v_\beta)^{1/p}] \le \alpha.$$

This means

$$m_\alpha \le n_\beta + [\lambda n_\beta];$$

hence

$$\begin{aligned}
\limsup_{\alpha\to 0_+}(\alpha m_\alpha) &\le \limsup_{\beta\to 0_+}\left[\beta(n_\beta - [\lambda n_\beta])^{(p'+1)/p'}\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right)^{(p'+1)/p'}\right] \\
&\quad + \limsup_{\beta\to 0_+}\left[\beta(n_\beta - [\lambda n_\beta])^{(p+1)/p}\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right)^{(p+1)/p}\right] \\
&\le \limsup_{\beta\to 0_+}\left[\beta(n_\beta - [\lambda n_\beta])^{(p'+1)/p'}\right]\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right)^{(p'+1)/p'} \\
&\quad + \limsup_{\beta\to 0_+}\left[\beta\left(n_\beta - [\lambda n_\beta]\right)^{(p+1)/p}\right]\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right)^{(p+1)/p} \\
&\le c(p,p')\|u'\|_{p'/(p'+1),(a,b)}\left(\frac{1+\lambda}{1-\lambda}\right)^{(p'+1)/p'} \\
&\quad + c(p,p')\|v'\|_{p/(p+1),(a,b)}\left(\frac{1+\lambda}{1-\lambda}\right)^{(p+1)/p}.
\end{aligned}$$

Since λ may be chosen arbitrarily small, we obtain

$$\limsup_{\alpha\to 0_+} \alpha m_\alpha \le c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)}),$$

as asserted. □

Next we obtain a remainder estimate which improves the result of Theorem 6.4.

Theorem 7.2. *Let* $-\infty \le a < b \le \infty$, *let* $u \in L_{p'}(a,b)$, $v \in L_p(a,b)$ *and suppose that* $u' \in L_{p'/(p'+1)}(a,b) \cap C([a,b])$, $v' \in L_{p/(p+1)}(a,b) \cap C([a,b])$. *Then*

$$\limsup_{N\to\infty} \left| \gamma_p \int_a^b |u(t)v(t)|dt - \varepsilon_N N \right| N^{1/2}$$

$$\le c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)}) \left(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}\right) + 3\gamma_p \|uv\|_{1,(a,b)},$$

where γ_p *is as in Theorem 5.8,* ε_N *as in Lemma 6.14 and* $c(p,p')$ *is a constant depending only on* p *and* p'.

Proof. Let $\alpha > 0$. Then (see (7.1) and (7.2)) there are $m_\alpha \in \mathbb{N}$ and step-functions u_α, v_α such that

$$\|u_\alpha - u\|_{p',(a,b)} < \alpha, \qquad \|v_\alpha - v\|_{p,(a,b)} < \alpha;$$

and $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a corresponding family of non-overlapping intervals which cover (a,b). Plainly,

$$\left|\int_a^b |uv| - |u_\alpha v_\alpha| dt\right| \le \left|\int_a^b (uv - u_\alpha v_\alpha) dt\right| \le \alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)} + \alpha). \quad (7.5)$$

Let $N > 1$ be such that the corresponding $\varepsilon_N > 0$ from Lemma 6.14 is small and let $\{I_i\}_{i=1}^N$ be the non-overlapping intervals which occur in the definition of ε_N.

Put $J_1 = \{j; I_i \subset w_\alpha(j) \text{ for some } i\}$, $J_2 = \{j; w_\alpha(j) \subset I_i \text{ for some } i\}$, $J_3 = \{j; w_\alpha(j) \not\subset I_i \not\subset w_\alpha(j), \text{ for all } i\}$, $L_1 = \{i; I_i \subset w_\alpha(j) \text{ for some } j\}$ and $L_2 = \{i; \text{ for all } j, I_i \not\subset w_\alpha(j)\}$. Then we see from Lemma 6.8 that

$$\begin{aligned}\gamma_p \int_a^b u_\alpha v_\alpha dt &= \gamma_p \left(\sum_{j\in J_1} + \sum_{j\in J_2} + \sum_{j\in J_3} \right) \xi_j \psi_j |w_\alpha(j)| \\ &\le \sum_{i\in L_1} \mathscr{A}(I_i, u_\alpha, v_\alpha) + 2\sum_{i\in L_2} \mathscr{A}(I_i, u_\alpha, v_\alpha) \\ &\quad + \sum_{j\in J_2} \gamma_p \xi_j \psi_j |w_\alpha(j)|. \end{aligned} \quad (7.6)$$

Lemmas 6.9, 6.10 as well as the estimates

$$\begin{aligned}\gamma_p \xi_j \psi_j |w_\alpha(j)| &\le \mathscr{A}(w_\alpha(j), u_\alpha, v_\alpha) \\ &\le \mathscr{A}(w_\alpha(j), u, v) + \|u - u_\alpha\|_{p',w_\alpha(j)} \|v - v_\alpha\|_{p,w_\alpha(j)} \\ &\quad + \|u\|_{p',w_\alpha(j)} \|v - v_\alpha\|_{p,w_\alpha(j)} \\ &\quad + \|u - u_\alpha\|_{p',w_\alpha(j)} \|v\|_{p,w_\alpha(j)}\end{aligned}$$

and $\mathscr{A}(w_\alpha(j),u,v) \le \mathscr{A}(I_i,u,v) \le \varepsilon_N$ for $w_\alpha(j) \subset I_i$ now show that the right-hand side of (7.6) may be estimated from above by

$$\begin{aligned}\sum_{I_i\subset w_\alpha(j)} \mathscr{A}(I_i,u,v) + 2\sum_{I_i\not\subset w_\alpha(j)} \mathscr{A}(I_i,u,v) + \varepsilon m_\alpha \\ +3\sum_{i=1}^{N}(\|u-u_\alpha\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} + \|u\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} \\ +\|u-u_\alpha\|_{p',I_i}\|v\|_{p,I_i}).\end{aligned} \tag{7.7}$$

To proceed further, note that $\mathscr{A}(I_i,u,v) \le \varepsilon_N$,

$$\#\{i : I_i \subset w_\alpha(j) \text{ for some } j\} \le N$$

and

$$\#\{i : \text{ for all } j, I_i \not\subset w_\alpha(j)\} \le m_\alpha.$$

It follows that

$$\begin{aligned}\gamma_p \int_a^b u_\alpha v_\alpha &\le N\varepsilon_N + 3m_\alpha\varepsilon_N \\ &\quad +3\sum_{i=1}^{N}(\|u-u_\alpha\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} + \|u\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} \\ &\quad +\|u-u_\alpha\|_{p',I_i}\|v\|_{p,I_i}) \\ &\le N\varepsilon_N + 3m_\alpha\varepsilon_N + 2\alpha^2 + 2\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}).\end{aligned} \tag{7.8}$$

On the other hand, since $\mathscr{A}(I_i,u,v) = \varepsilon_N$ for $2 \le i \le N$ and $N - 2m_\alpha \le \#\{i; I_i \subset w_\alpha(j) \text{ for some } j\}$, we see that

$$\begin{aligned}(N-2m_\alpha-1)\varepsilon_N &\le \sum_{I_i\subset w_\alpha(j)} \mathscr{A}(I_i,u,v) \\ &= \sum_{I_i\subset w_\alpha(j)} \mathscr{A}(I_i,u_\alpha,v_\alpha) \\ &\quad + \sum_{I_i\subset w_\alpha(j)} [\mathscr{A}(I_i,u,v) - \mathscr{A}(I_i,u_\alpha,v_\alpha)] \\ &\le \sum_{I_i\subset w_\alpha(j)} \gamma_p|I_i||\xi_j||\psi_j| \\ &\quad + \sum_{I_i\subset w_\alpha(j)} (\|u-u_\alpha\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} \\ &\quad +\|u\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} + \|u-u_\alpha\|_{p',I_i}\|v\|_{p,I_i}) \\ &\le \gamma_p\int_a^b |u_\alpha v_\alpha|dt + \alpha^2 + \alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}) \\ &\le \gamma_p\int_a^b |uv|dt + 2\alpha^2 \\ &\quad +2\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}),\end{aligned} \tag{7.9}$$

the final inequality following from (7.5). Together with (7.8) and (7.5) this shows that

$$\begin{aligned} \varepsilon_N(N - 2m_\alpha - 1) - 2\alpha^2 - 2\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}) \\ \leq \gamma_p \int_a^b |uv|dt \\ \leq \varepsilon_N(N + 3m_\alpha) + 3\alpha^2 + 3\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}). \end{aligned} \tag{7.10}$$

From Lemma 7.1 we can see that for any $N > 1$, we can find $\alpha > 0$ such that $m_\alpha \geq [N^{1/2}] \geq m_\alpha - 2$. Then (7.9) gives

$$\begin{aligned} N^{1/2}|\gamma_p \int_a^b |uv|dt - N\varepsilon_N| \leq 3N\varepsilon_N + 3\alpha^2(N^{1/2} - 1) \\ +3\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)})N^{1/2}. \end{aligned}$$

Let $N \to \infty$; then $\varepsilon_N \to 0_+$ and $m_\alpha \leq N^{1/2} + 2 \to \infty$ and so $\alpha \to 0_+$. Hence

$$\begin{aligned} \limsup_{N\to\infty} N^{1/2}|\gamma_p \int_a^b |uv|dt - N\varepsilon_N| \\ \leq 3\limsup_{N\to\infty} N\varepsilon_N + 3\limsup_{N\to\infty} \alpha^2 N^{1/2} \\ +3\limsup_{N\to\infty} \alpha N^{1/2}(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}). \end{aligned}$$

Since $\lim_{N\to\infty} \varepsilon_N N = \gamma_p \int_a^b |uv|dt$, by Theorem 6.3, we finally see, with the help of Lemma 7.1, that

$$\begin{aligned} \limsup_{N\to\infty} \quad N^{1/2}|\gamma_p \int_a^b |uv| - N\varepsilon| \\ \leq 3\gamma_p \int_a^b |uv|dt \\ +3c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)})(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}), \end{aligned}$$

as required. □

Armed with this result it is now easy to give the promised remainder estimate for the strict numbers of $T : L_p(a,b) \to L_p(a,b)$.

Theorem 7.3. *Let* $-\infty \leq a < b \leq \infty$, *suppose that* $u \in L_{p'}(a,b)$, $v \in L_p(a,b)$ *and let* $u' \in L_{p'/(p'+1)}(a,b) \cap C((a,b))$, $v' \in L_{p/(p+1)}(a,b) \cap C((a,b))$. *Then*

$$\begin{aligned} \limsup_{n\to\infty} n^{1/2}\left|\gamma_p \int_a^b |uv|dt - n\widetilde{s}_n(T)\right| \leq 3\gamma_p \int_a^b |uv|dt \\ +3c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)})(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}), \end{aligned}$$

where γ_p *is as in Theorem 5.8 and* $\widetilde{s}_n(T)$ *stands for any* n*-th strict* s*-number of* T.

Proof. Simply use Theorems 6.2, 6.3, 7.2, Lemma 6.14 and the fact that

$$\lim_{n\to\infty} n^{1/2} a_n(T) = 0.$$

□

If the interval (a,b) is bounded, it follows immediately from Hölder's inequality that Theorem 7.3 gives rise to

Theorem 7.4. *Let $-\infty < a < b < \infty$ and suppose that $u', v' \in C([a,b])$. Then*

$$\limsup_{n\to\infty} n^{1/2} |\gamma_p \int_a^b |uv|dt - n\widetilde{s}_n(T)|$$

$$\leq 3\gamma_p \int_a^b |uv|dt + 3c(p,p')(b-a)(\|u'\|_{p',(a,b)} + \|v'\|_{p,(a,b)})(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}).$$

From the following observation we can see that any optimal exponent of n in Theorem 7.3 has to belong to $[1/2, 1]$.

Remark 7.1. Let $-\infty \leq a < b \leq \infty$ with γ_p is as in Theorem 5.8.

(i) Let $\alpha < 1/2$. Then for every $u \in L_{p'}(a,b)$, $v \in L_p(a,b)$ with

$$u' \in L_{p'/(p'+1)}(a,b) \cap C([a,b]), \quad v' \in L_{p/(p+1)}(a,b) \cap C([a,b])$$

we have

$$\limsup_{n\to\infty} n^{\alpha} \left| \gamma_p \int_a^b |uv|dt - n\widetilde{s}_n(T) \right| = 0.$$

(ii) Let $\alpha > 1$. Then there exist a and b, and functions u and v satisfying the conditions of Theorem 4.2 on the interval defined by a and b, such that

$$\limsup_{n\to\infty} n^{\alpha} \left| \gamma_p \int_a^b |uv|dt - n\widetilde{s}_n(T) \right| = \infty.$$

Proof. (i) This follows from (7.10) on putting $m_\alpha = [N^\alpha]$ or $[N^\alpha] + 1$.

(ii) Take $(a,b) = (0,1)$ and $u = 1$, $v = 1 + x$. Then from (7.10), with $m_\alpha = [N^\alpha]$ a lower bound results which is unbounded as $\varepsilon_N \to 0$ and the claim follows. □

7.2 The Second Asymptotic Term

In this section we shall obtain a bound for the second asymptotic term. To do so conditions on u'/u and v'/v must be introduced as we shall see.

We start with the following obvious lemma about $\mathscr{A}(I)$.

Lemma 7.5. *Let $I = (c,d) \subset (a,b)$; let $|u_1| \geq |u_2| > 0$ and $|v_1| \geq |v_2| > 0$. Then*

$$\mathscr{A}(I, u_1, v_1) \geq \mathscr{A}(I, u_2, v_2) \geq 0.$$

Now we are ready to prove the following lemma about the behaviour of $\varepsilon N(\varepsilon)$.

Lemma 7.6. *Let* $1 < p < \infty$, $I = (a,b)$, $u \in L_{p'}(I)$, $v \in L_p(I)$ *and* $(v'/v), (u'/u) \in L_1(I) \cap C[a,b]$. *Then*

$$\lim_{N\to\infty} \left| N \left[\varepsilon_N N - \gamma_p \int_I |u(x)v(x)|dx \right] \right| \\ \leq \int_I u(x)v(x)dx \left[\int_I \frac{v'(x)}{v(x)}dx + \int_I \frac{u'(x)}{u(x)}dx + \gamma_p \right. \\ \left. + \left(\int_I \frac{u'(x)}{u(x)}dx \right) \left(\int_I \frac{v'(x)}{v(x)}dx \right) \right],$$

where γ_p *is as in Theorem 5.8 and* ε_N *is defined as in Lemma 6.14.*

Proof. Take $N > 1$ and set $\varepsilon := \varepsilon_N$. Let us recall that when $N \to \infty$ then $\varepsilon_N \to 0_+$ and vice versa. Consider N so large that $\|T\| > \varepsilon > 0$. Then according to Lemma 6.14 we have the following partition: $I = \cup_{i=1}^N I_i$, $\mathscr{A}(I_i) = \varepsilon$ for $i = \{2,...,N\}$ and $\mathscr{A}(I_1) < \varepsilon$. Define the following step functions:

$$u^{+,\varepsilon}(x) = \sum_{i=1}^N u_i^{+,\varepsilon} \chi_{I_i}(x), \qquad v^{+,\varepsilon}(x) = \sum_{i=1}^N v_i^{+,\varepsilon} \chi_{I_i}(x)$$
$$u^{-,\varepsilon}(x) = \sum_{i=1}^N u_i^{-,\varepsilon} \chi_{I_i}(x), \qquad v^{-,\varepsilon}(x) = \sum_{i=1}^N v_i^{-,\varepsilon} \chi_{I_i}(x)$$

where

$$u_i^{+,\varepsilon} = \sup_{x\in I_i} |u(x)|, \qquad u_i^{-,\varepsilon} = \inf_{x\in I_i} |u(x)|$$
$$v_i^{+,\varepsilon} = \sup_{x\in I_i} |v(x)|, \qquad v_i^{-,\varepsilon} = \inf_{x\in I_i} |v(x)|.$$

Then we have from Lemma 7.5:

$$\gamma_p u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| \leq \mathscr{A}(I_i) \leq \gamma_p u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i|, \tag{7.11}$$

and we can see that

$$\int_I u^{-,\varepsilon}(x)v^{-,\varepsilon}(x)dx \leq \int_I |u(x)v(x)|dx \leq \int_I u^{+,\varepsilon}(x)v^{+,\varepsilon}(x)dx.$$

Now we estimate the following quantity from above:

$$\begin{aligned} K(\varepsilon) &:= \int_I (u^{+,\varepsilon}(x)v^{+,\varepsilon}(x) - u^{-,\varepsilon}(x)v^{-,\varepsilon}(x))dx \\ &= \sum_{i=1}^N |I_i|(u_i^{+,\varepsilon} v_i^{+,\varepsilon} - u_i^{-,\varepsilon} v_i^{-,\varepsilon}) \\ &= \sum_{i=1}^N |I_i|(u_i^{+,\varepsilon} v_i^{+,\varepsilon} - u_i^{+,\varepsilon} v_i^{-,\varepsilon} + u_i^{+,\varepsilon} v_i^{-,\varepsilon} - u_i^{-,\varepsilon} v_i^{-,\varepsilon}). \end{aligned}$$

Denote $u_i = |I_i| \max_{x\in I_i} |u'(x)|$ and $v_i = |I_i| \max_{x\in I_i} |v'(x)|$.
With the use of $(v_i^{+,\varepsilon} - v_i^{-,\varepsilon}) \leq v_i$ and $(u_i^{+,\varepsilon} - u_i^{-,\varepsilon}) \leq u_i$ we have

$$\begin{aligned}
K(\varepsilon) &\leq \sum_{i=1}^{N} |I_i| \left[u_i^{+,\varepsilon} v_i + v_i^{-,\varepsilon} u_i \right] \\
&\qquad\qquad \left(\text{use } \gamma_p |I_i| u_i^{-,\varepsilon} v_i^{-,\varepsilon} \leq \mathscr{A}(I_i) \leq \varepsilon \right) \\
&\leq \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \left[\frac{u_i^{+,\varepsilon}}{u_i^{-,\varepsilon}} \frac{v_i}{v_i^{-,\varepsilon}} + \frac{v_i^{-,\varepsilon}}{v_i^{-,\varepsilon}} \frac{u_i}{u_i^{-,\varepsilon}} \right] \\
&\leq \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \frac{u_i^{-,\varepsilon} + u_i}{u_i^{+,\varepsilon}} . \frac{v_i}{v_i^{-,\varepsilon}} + \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \frac{u_i}{u_i^{-,\varepsilon}} \\
&\leq \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \left[1 + \frac{u_i}{u_i^{+,\varepsilon}} \right] \frac{v_i}{v_i^{-,\varepsilon}} + \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \frac{u_i}{u_i^{-,\varepsilon}} \\
&\leq \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \left[1 + \sum_{i=1}^{N} \frac{u_i}{u_i^{-,\varepsilon}} \right] \frac{v_i}{v_i^{-,\varepsilon}} + \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \frac{u_i}{u_i^{-,\varepsilon}} \\
&= \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \frac{u_i}{u_i^{-,\varepsilon}} + \frac{\varepsilon}{\gamma_p} \sum_{i=1}^{N} \frac{v_i}{v_i^{-,\varepsilon}} \\
&\qquad + \frac{\varepsilon}{\gamma_p} \left(\sum_{i=1}^{N} \frac{v_i}{v_i^{-,\varepsilon}} \right) \left(\sum_{i=1}^{N} \frac{u_i}{u_i^{-,\varepsilon}} \right).
\end{aligned}$$

From (7.11) we have:

$$\sum_{i=1}^{N} \gamma_p u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| \leq \varepsilon N \qquad \text{and} \qquad \sum_{i=1}^{N} \gamma_p u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i| \geq \varepsilon (N-1)$$

and then

$$\begin{aligned}
\sum_{i=1}^{N} \gamma_p u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| - \gamma_p \int_I |uv| \, dx &\leq \varepsilon N - \gamma_p \int_I uv \, dx \\
&\leq \sum_{i=1}^{N} \gamma_p u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i| + \varepsilon - \gamma_p \int_I uv \, dx
\end{aligned}$$

which gives

$$-K(\varepsilon) \leq \varepsilon N - \gamma_p \int_I |uv| \, dx \leq K(\varepsilon) + \varepsilon$$

and

$$-NK(\varepsilon) \leq N \left(\varepsilon N - \gamma_p \int_I |uv| \, dx \right) \leq NK(\varepsilon) + \varepsilon N.$$

Using $\lim_{N\to\infty} \varepsilon_N N = \lim_{\varepsilon_N \to 0_+} \varepsilon_N N = \gamma_p \int_I |uv|\, dx$ and

$$\lim_{N\to\infty} \frac{K(\varepsilon_N)}{\varepsilon_N} = \frac{1}{\gamma_p}\int_I \left|\frac{u'}{u}\right| + \frac{1}{\gamma_p}\int_I \left|\frac{v'}{v}\right| + \frac{1}{\gamma_p}\int_I \left|\frac{u'}{u}\right| \int_I \left|\frac{v'}{v}\right|$$

we obtain:

$$\begin{aligned}\limsup_{N\to\infty} & \left| N\left(\varepsilon_N N - \gamma_p \int_I |uv|\, dx\right)\right| \\ & \le \left(\int_I |uv|\right)\left(\int_I \left|\frac{u'}{u}\right| + \int_I \left|\frac{v'}{v}\right| + \int_I \left|\frac{u'}{u}\right| \int_I \left|\frac{v'}{v}\right|\right) + \gamma_p \int_I |uv|.\end{aligned}$$

□

The following theorem give us a bound for the second asymptotic.

Theorem 7.5. *Let $1 < p < \infty$, $-\infty \le a < b \le \infty$ and $I = (a,b)$, let $u \in L_{p'}(I)$, $v \in L_p(I)$ and $(v'/v), (u'/u) \in L_1(I) \cap C[a,b]$. Then*

$$\begin{aligned}\limsup_{n\to\infty} & \left| n\left[n\widetilde{s}_n(T) - \gamma_p \int_I |u(x)v(x)|dx\right]\right| \\ & \le \int_I |u(x)v(x)|dx \left[\int_I \left|\frac{v'(x)}{v(x)}\right| dx + \int_I \left|\frac{u'(x)}{u(x)}\right| dx + \gamma_p \right. \\ & \qquad \left. + \left(\int_I \left|\frac{u'(x)}{u(x)}\right| dx\right)\left(\int_I \left|\frac{v'(x)}{v(x)}\right| dx\right)\right],\end{aligned}$$

where $\widetilde{s}_n(T)$ stands for any strict s-number of the Hardy-type operator T.

Proof. From Theorem 6.2 it follows that for each $N > 1$ we have $\varepsilon_N = \tilde{s}_n(T)$, which concludes the proof. □

Theorem 7.5 gives the following information about the second asymptotic:

$$\widetilde{s}_n(T) = \frac{1}{n}\gamma_p \int_I |u(x)v(x)|dx + O(n^{-2}).$$

7.3 l_q and Weak-l_q Estimates when $1 < p < \infty$

In this section we start by introducing a sequence which helps us to generalize Theorem 6.4 and then we show that on sequence spaces l_q and weak-l_q the behaviour of this sequence is similar to the behaviour of a sequence generated by strict s-numbers.

With $U(x) := \int_a^x |u(t)|^{p'} dt$, we define $\xi_k \in \mathbb{R}^+$ by,

$$U(\xi_k) = 2^{kp'/p}; \tag{7.12}$$

if $u \notin L_{p'}(a,b)$, k may be any integer, but if $u \in L_{p'}(a,b), 2^{kp'/p} \leq \|u\|_{p'}$. We shall refer to the range of possible values of k as being admissible. For each admissible k we set

$$\sigma_k := \|U^{1/p'}v\|_{p,Z_k}, \qquad Z_k = (\xi_k, \xi_{k+1}), \tag{7.13}$$

so that

$$2^{k/p}\|v\|_{p,Z_k} \leq \sigma_k \leq 2^{(k+1)/p}\|v\|_{p,Z_k}. \tag{7.14}$$

For non–admissible k we set $\sigma_k = 0$. The sequence $\{\sigma_k\}$ is the analogue of that defined in [2, Sect. 3], which in turn was motivated by a similar sequence introduced in [5].

We recall Jensen's inequality (see, for example [82], p.133) which will be of help in the next lemma.

Theorem 7.6. *If F is a convex function, and $h(.) \geq 0$ is a function such that $\int_I h(t)dt = 1$, then for every non-negative function g,*

$$F(\int_I h(t)g(t)dt) \leq \int_I h(t)F(g(t))dt.$$

The following technical lemma has a central role in this section.

Lemma 7.7. *Let $k_0, k_1, k_2 \in \mathbb{Z}$ with $k_0 \leq k_1 \leq k_2$, $k_0 < k_2$, and let $I_j = (a_j, b_j)$ $(j = 0, 1, \ldots, l)$ be non–overlapping intervals in (a,b) which are such that $I_j \subset Z_{k_2}$ $(j = 1, \ldots, l)$, $a_0 \in Z_{k_0}$, $b_0 \in Z_{k_2}$. Let $x_j \in I_j$ $(j = 0, 1, \ldots, l)$ and $x_0 \in Z_{k_1}$. Then, if $\alpha \geq 1$,*

$$\begin{aligned} S &:= \sum_{j=0}^{l} \left(\int_{a_j}^{x_j} |u(t)|^{p'} dt \right)^{\alpha/p'} \|v\|^{\alpha}_{p,(x_j,b_j)} \\ &\leq 2^{\alpha/p}(2^{\alpha/p}+1) \max_{k_1 \leq n \leq k_2} \sigma_n^{\alpha}. \end{aligned} \tag{7.15}$$

Proof. On using Jensen's inequality, we have

$$\begin{aligned} S &\leq \left(\int_{\xi_{k_0}}^{\xi_{k_1+1}} |u(t)|^{p'} dt \right)^{\alpha/p'} \|v\|^{\alpha}_{p,(\xi_{k_1},\xi_{k_2+1})} + \sum_{j=1}^{l} \|u\|^{\alpha}_{p',I_j} \|v\|^{\alpha}_{p,I_j} \\ &\leq \left(2^{(k_1+1)p'/p} - 2^{k_0 p'/p} \right)^{\alpha/p'} \left(\sum_{n=k_1}^{k_2} \frac{\sigma_n^p}{2^n} \right)^{\alpha/p} + \|u(t)\|^{\alpha}_{p',Z_{k_2}} \|v\|^{\alpha}_{p,Z_{k_2}} \\ &\quad (\text{ by (7.14) }) \\ &\leq 2^{(k_1+1)\alpha/p} \left(2^{1-k_1} \max_{k_1 \leq n \leq k_2} \sigma_n^p \right)^{\alpha/p} + \left\{ 2^{(k_2+1)/p} \frac{\sigma_{k_2}}{2^{k_2/p}} \right\}^{\alpha} \\ &\leq \left\{ 2^{2/p} \max_{k_1 \leq n \leq k_2} \sigma_n \right\}^{\alpha} + \left\{ 2^{(k_2+1)/p} \frac{\sigma_{k_2}}{2^{k_2/p}} \right\}^{\alpha}, \end{aligned}$$

whence (7.15). □

Lemma 7.8. *The quantity $\mathscr{J}(a,b)$ defined in (6.4) satisfies*

$$\mathscr{J}(a,b) \le 2^{1/p}(2^{1/p}+1)\sup_k \sigma_k \le 2^{2/p}(2^{1/p}+1)\mathscr{J}(a,b). \tag{7.16}$$

Proof. From (6.4) and Lemma 7.7 with $\alpha = 1$,

$$\mathscr{J}(a,b) \le 2^{1/p}(2^{1/p}+1)\sup_k \sigma_k.$$

Also

$$\begin{aligned} \sigma_k &\le 2^{(k+1)/p}\|v\|_{p,Z_k} \\ &\le \frac{2^{(k+1)/p}}{U(\xi_k)^{1/p'}}\|u\|_{p',(a,\xi_k)}\|v\|_{p,(\xi_k,b)} \\ &\le 2^{1/p}\mathscr{J}(a,b). \end{aligned}$$

□

Corollary 7.1. *The operator $T : L_p(a,b) \to L_p(a,b)$ is bounded if and only if the sequence $\{\sigma_k\} \in l_\infty$, in which case the norm of T and that of the sequence are equivalent:*

$$\|T\| \asymp \|\{\sigma_k\}\|_\infty. \tag{7.17}$$

Also T is compact if and only if $\lim_{k\to\pm\infty}\sigma_k = 0$.

Proof. The first part is an immediate consequence of Theorem 4.4 and Lemma 7.8. We also have from Lemma 7.8, as in the proof of Lemma 7.8,

$$\mathscr{J}(a,\xi_{k_2}) \le 2^{1/p}(2^{1/p}+1)\max_{n\le k_2}\sigma_n \le 2^{2/p}(2^{1/p}+1)\mathscr{J}(a,\xi_{k_2+1})$$

and

$$\mathscr{J}(\xi_{k_0},b) \le 2^{1/p}(2^{1/p}+1)\max_{n\ge k_0}\sigma_n \le 2^{2/p}(2^{1/p}+1)\mathscr{J}(\xi_{k_0-1},b).$$

Since $\xi_{k_2} \to a$ if and only if $k_2 \to -\infty$, and $\xi_{k_0} \to b$ if and only if $k_0 \to \infty$ in the case $u \notin L_{p'}(a,b)$ and otherwise to the largest admissible value of k in the definition of σ_k, the corollary follows. □

The main result is

Theorem 7.7. *Suppose that $1 < p < \infty$ and (6.2) is satisfied, and that $\sum_{n\in\mathbb{Z}}\sigma_n$ is convergent. Then*

$$\liminf_{n\to\infty} n\rho_n(T) = \gamma_p \int_a^b |u(t)v(t)|dt, \tag{7.18}$$

where $\rho_n(T)$ stands for any strict s-number and γ_p is as in Theorem 5.8.

Proof. Let $I = [c,d] \subset (a,b)$ and suppose that $c \in [\xi_{k_0}, \xi_{k_0+1}]$, $d \in [\xi_{k_1}, \xi_{k_1+1})$. Take $\varepsilon > 0$ sufficiently small. Then according to Remark 6.2 and Lemma 6.11 there exist $M(\varepsilon) > 3$ and a sequence $a = a_0 < a_1 < ... < a_{M(\varepsilon)} = b$ such $\mathscr{A}(a_{i-1}, a_i) = \varepsilon$ for $i = 2, ..., M(\varepsilon) - 1$, $\|T_{a,(a_0,a_1)}\| = \varepsilon$ and $\mathscr{A}(a_{M(\varepsilon)-1}, a_{M(\varepsilon)}) \leq \varepsilon$. Define $I_j(\varepsilon) = (a_{j-1}, a_j)$, $j = 1, 2, ..., M(\varepsilon)$, forming the covering of (a,b).

For the above ε we define, according to Remark 6.2 and Lemma 6.11, a partition of I into $M(I,\varepsilon)$ subintervals $\{J_i\}_{i=1}^{M(I,\varepsilon)}$ such that $\|T_{c,J_1}\| = \varepsilon$, $\mathscr{A}(J_i) = \varepsilon$ for $i = 2, ..., M(I,\varepsilon) - 1$, and $\mathscr{A}(J_{M(I,\varepsilon)}) \leq \varepsilon$.

Set

$$m_0(\varepsilon) = \#\{j : I_j(\varepsilon) \subset [a,c]\}$$
$$m_1(\varepsilon) = \#\{j : I_j(\varepsilon) \subset [a,d]\}.$$

Then

$$m_1(\varepsilon) - m_0(\varepsilon) \leq M(I,\varepsilon) + 1$$

and

$$\begin{aligned}
\frac{\varepsilon}{2}\big(M(\varepsilon) - M(I,\varepsilon) - 9\big) &\leq \varepsilon\left(\left[\frac{m_0(\varepsilon)}{2}\right] + \left[\frac{M(\varepsilon)}{2}\right] - \left[\frac{m_1(\varepsilon)}{2}\right] - 2\right) \\
&\leq \sum_{j=1}^{[m_0(\varepsilon)/2]} \mathscr{A}(I_{2j-1} \cup I_{2j}; u, v) + \sum_{j=[m_1(\varepsilon)/2]+2}^{[M(\varepsilon)/2]} \mathscr{A}(I_{2j-1} \cup I_{2j}; u, v) \\
&\leq \sum_{j=1}^{[m_0(\varepsilon)/2]} \mathscr{J}(I_{2j-1} \cup I_{2j}; u, v) + \sum_{j=[m_1(\varepsilon)/2]+2}^{[M(\varepsilon)/2]} \mathscr{J}(I_{2j-1} \cup I_{2j}; u, v) \\
&\leq 3\sum_{n \leq k_0} \sigma_n + 3\sum_{n \geq k_1} \sigma_n,
\end{aligned}$$

on using (6.8) and (7.16).

It follows from Theorem 6.3 that

$$\limsup_{\varepsilon \to 0_+} \varepsilon M(\varepsilon) \leq \gamma_p \int_{\xi_{k_0}}^{\xi_{k_1+1}} |u(t)v(t)|dt + 3\left(\sum_{n \leq k_0} \sigma_n + \sum_{n \geq k_1} \sigma_n\right),$$

which yields

$$\limsup_{\varepsilon \to 0_+} \varepsilon M(\varepsilon) \leq \gamma_p \int_a^b |u(t)v(t)|dt.$$

On setting $n = M(\varepsilon) + 1$ in Theorem 6.2, we get $\varepsilon \geq a_n(T)$ and hence

$$\limsup_{n \to \infty} n\, a_n(T) \leq \gamma_p \int_a^b |u(t)v(t)|dt.$$

Similarly, from Theorem 6.3,

$$\liminf_{\varepsilon \to 0_+} \varepsilon M(\varepsilon) \geq \gamma_p \int_a^b |u(t)v(t)|dt,$$

and from Theorem 6.2,

$$\liminf_{n \to \infty} n\, i_n(T) \geq \gamma_p \int_a^b |u(t)v(t)|dt.$$

□

Next we show that the sequences $\{\rho_n(T)\}_{n\in\mathbb{N}}, \{\sigma_n\}_{n\in\mathbb{Z}}$, where $\rho_n(T)$ stands for any strict s-number, belong to l_q and weak-l_q sequence spaces with the same exponent q, and have equivalent norms. We first need some preparatory results.

Lemma 7.9. *Let $I = [c,d] \subset (a,b)$ and, for $\varepsilon > 0$, suppose that*

$$\sigma(\varepsilon) := \{k \in \mathbb{Z} : Z_k \subset I, \sigma_k > \varepsilon\}$$

has at least 4 distinct elements. Then $\mathscr{A}(I) > \varepsilon/8$.

Proof. Let $Z_{k_i}, i = 1,2,3,4$, $k_1 < k_2 < k_3 < k_4$, be 4 distinct members of $\sigma(\varepsilon)$, and set $I_1 = (\xi_{k_1}, \xi_{k_2})$, $I_2 = (\xi_{k_2+1}, \xi_{k_4})$. Then, with $f_0 = \chi_{I_1} + \chi_{I_2}$,

$$\begin{aligned}
\mathscr{A}(I) &\geq \inf_\alpha \|v(x)\left(\int_c^x |u(t)|f_0(t)dt - \alpha\right)\|_{p,I} \\
&\geq \inf_\alpha \max\left\{\|v\|_{p,Z_{k_2}}\left|\int_{I_1}|u(t)|dt - \alpha\right|, \|v\|_{p,Z_{k_4}}\left|\int_{I_1\cup I_2}|u(t)|dt - \alpha\right|\right\} \\
&= \inf_\alpha \max\Big\{\|v\|_{p,Z_{k_2}}|2^{k_2p'/p} - 2^{k_1p'/p} - \alpha|, \\
&\qquad\qquad \|v\|_{p,Z_{k_4}}|2^{k_2p'/p} - 2^{k_1p'/p} + 2^{k_4p'/p} - 2^{(k_2+1)p'/p} - \alpha|\Big\} \\
&\geq \inf_\alpha \max\Big\{\frac{\varepsilon}{2^{(k_2+1)p'/p}}|2^{k_2p'/p} - 2^{k_1p'/p} - \alpha|, \\
&\qquad\qquad \frac{\varepsilon}{2^{(k_4+1)p'/p)}}|2^{k_2p'/p} - 2^{k_1p'/p} + 2^{k_4p'/p} - 2^{(k_2+1)p'/p} - \alpha|\Big\} \\
&\geq \frac{\varepsilon}{2^{k_4p'/p+1}}\frac{1}{2}\left(2^{k_4p'/p} - 2^{(k_2+1)p'/p}\right) \geq \frac{\varepsilon}{8}.
\end{aligned}$$

□

Lemma 7.10. *Let $\varepsilon > 0$ and $M(\varepsilon) = M((a,b),\varepsilon)$, where $M((a,b),\varepsilon)$ was introduced in the proof of Theorem 7.7. Then*

$$\#\{k \in \mathbb{Z} : \sigma_k > 8\varepsilon\} \leq 5M(\varepsilon) + 3. \tag{7.19}$$

Proof. Clearly, with $I_i = (a_{i-1}, a_i)$ the intervals obtained from Remark 6.2 and Lemma 6.11 when $I = (a, b)$,

$$\#\{k \in \mathbb{Z} : c_i \in \bar{Z}_k \text{ for some } i \in \{1, 2, \ldots, M(\varepsilon)\}\} \leq 2M(\varepsilon).$$

Also, for every $k \in \mathbb{Z}$ not included in the above set, $\bar{Z}_k \subset I_i$ for some $i \in \{1, 2, \ldots, M(\varepsilon)\}$. Hence, by Lemma 7.9,

$$\begin{aligned}\#\{k \in \mathbb{Z} : \sigma_k > 8\varepsilon\} &\leq 2M(\varepsilon) + 3(M(\varepsilon) + 1) \\ &= 5M(\varepsilon) + 3.\end{aligned}$$

□

Lemma 7.11. *For all $t > 0$,*

$$\#\{k \in \mathbb{Z} : \sigma_k > t\} \leq 10\#\{k \in \mathbb{N} : a_k(T) > t/8\} + 23. \tag{7.20}$$

Proof. By Theorem 6.2,

$$\#\{k \in \mathbb{N} : a_k(T) > \varepsilon\} \geq \frac{M(\varepsilon)}{2} - 2.$$

Hence, by Lemma 7.10,

$$\begin{aligned}\#\{k \in \mathbb{Z} : \sigma_k > t\} &\leq 5M(t/8) + 3 \\ &\leq \#\{k \in \mathbb{N} : a_k(T) > t/8\} + 23.\end{aligned}$$

□

Lemma 7.12. *For all $q > 0$,*

$$\|\{\sigma_k\}\|^q_{l_q(\mathbb{Z})} \leq 10\,(8^q)\|\{a_k(T)\}\|^q_{l_q(\mathbb{N})} + 23\|\{\sigma_k\}\|^q_{l_\infty(\mathbb{Z})}. \tag{7.21}$$

Proof. Let $\lambda = \|\{\sigma_k\}\|_{l_\infty(\mathbb{Z})}$. Then, by Lemma 7.11,

$$\begin{aligned}\|\{\sigma_k\}\|^q_{l_q(\mathbb{Z})} &= q\int_0^\lambda t^{q-1}\#\{k \in \mathbb{Z} : \sigma_k > t\}\,dt \\ &\leq 10\,q\int_0^\lambda t^{q-1}\#\{k \in \mathbb{N} : a_k(T) > t/8\}\,dt + 23\lambda^q \\ &\leq 10\,(8^q)\|\{a_k(T)\}\|^q_{l_q(\mathbb{N})} + 23\lambda^q.\end{aligned}$$

□

Corollary 7.2. *For any $q > 0$ there exists a constant $C > 0$ such that*

$$\|\{\sigma_k\}\|_{l_q(\mathbb{Z})} \leq C\|\{a_k(T)\}\|_{L_q(\mathbb{N})}. \tag{7.22}$$

Proof. By (7.17),

$$\begin{aligned}\|\{\sigma_k\}\|_{l_\infty(\mathbb{Z})} &\le C\|T\| = Ca_1(T)\\ &\le C\|\{a_k(T)\}\|_{l_q(\mathbb{N})}.\end{aligned}$$

The result then follows from Lemma 7.12. □

Theorem 7.8. *For $q \in (1,\infty)$, $\{a_k(T)\} \in l_q(\mathbb{N})$ if and only if $\{\sigma_k\} \in l_q(\mathbb{Z})$, and*

$$\|\{\sigma_k\}\|_{l_q(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l_q(\mathbb{N})}.$$

Proof. Define

$$N(I,\varepsilon) := \inf\{n : I = \cup_{i=1}^n I_i, \mathscr{J}(I_i) \le \varepsilon\}.$$

Since $\mathscr{J}(c,d) \le \|u\|_{p',(c,d)}\|v\|_{p,(c,d)}$ for any (c,d) and $\|.\|_p$ is absolutely continuous, then $N(I,\varepsilon) < \infty$. Let $I_i, i = 1,2,\ldots,N(\varepsilon)$, be the intervals in $N(I,\varepsilon)$ with $I = (a,b)$ and $N(\varepsilon) \equiv N((a,b),\varepsilon)$: note that in view of the compactness of T and continuity of $\mathscr{J}(.,.)$, we have $\mathscr{J}(I_i) = \varepsilon$. We group the intervals I_i into families $\mathbf{F}_j, j = 1,2,\ldots$ such that each $\mathbf{F}_j$ consists of the maximal number of those intervals satisfying the hypothesis of Lemma 7.7 : they lie within (ξ_{k_0}, ξ_{k_2+1}) for some k_0, k_2, and the next interval I_k intersects Z_{k_2+1}. Hence, by Lemma 7.7, there is a positive constant c such that

$$\varepsilon \# \mathbf{F}_j \le c \max_{k_0 \le n \le k_2} \sigma_n = c\sigma_{k_j},$$

say. It follows that, with $n_j = [c\sigma_{k_j}/\varepsilon]$,

$$\begin{aligned}N(\varepsilon) &= \sum_j \#\mathscr{F}_j\\ &\le \sum_j \sum_{n=1}^{n_j} 1 = \sum_{n=1}^{\infty} \sum_{j:n_j \ge n} 1\\ &= \sum_{n=1}^{\infty} \#\left\{j : \frac{c\sigma_{k_j}}{\varepsilon} \ge n\right\}\\ &\le \sum_{n=1}^{\infty} \#\left\{k : \sigma_k \ge \frac{n\varepsilon}{c}\right\}. \qquad (7.23)\end{aligned}$$

Thus, if $\{\sigma_k\} \in l_q(\mathbb{Z})$ for some $q \in (1,\infty)$,

$$\begin{aligned}q\int_0^\infty t^{q-1}N(t)dt &\le q\int_0^\infty \sum_{n=1}^{\infty} t^{q-1}\#\left\{k : \sigma_k > \frac{nt}{c}\right\}dt\\ &= qc^q \int_0^\infty \sum_{n=1}^{\infty} n^{-q}s^{q-1}\#\{k : \sigma_k > s\}\,ds\\ &\preceq \|\{\sigma_k\}\|_{l_q(\mathbb{Z})}^q \qquad (7.24)\end{aligned}$$

where $\preceq$ stands for less than or equal to a constant multiple of what follows. From Theorem 6.2, $a_{M(\varepsilon)+1}(T) \leq 2\varepsilon$ and so

$$\begin{aligned}\#\{k \in \mathbb{N} : a_k(T) > t\} &\leq M(t/2) + 1 \\ &\leq N(t/2) + 1.\end{aligned}$$

This yields

$$\begin{aligned}\|\{a_k(T)\}\|_{l_q(\mathbb{N})}^q &= q\int_0^\infty t^{q-1}\#\{k \in \mathbb{N} : a_k(T) > t\}\,dt \\ &\leq q\int_0^{\|T\|} t^{q-1}\left[N(\frac{t}{2}) + 1\right] dt \\ &\preceq \|\{\sigma_k\}\|_{l_q(\mathbb{Z})}^q + \|T\|^q \\ &\preceq \|\{\sigma_k\}\|_{l_q(\mathbb{Z})}^q\end{aligned}$$

by (7.24) and since $\|T\| \preceq \|\{\sigma_k(T)\}\|_{l_\infty(\mathbb{Z})} \leq \|\{\sigma_k\}\|_{l_q(\mathbb{Z})}$, by (7.17). The theorem follows from (7.22). □

The final result in this section concerns the weak l_q spaces, which we denote by $l_{q,\omega}$ ($l_{q,\infty}$ in the Lorentz scale). Recall that $l_{q,\omega}(\mathbb{Z})$ is the space of sequences $x = \{x_k\}$ such that

$$\|x\|_{l_{q,\omega}(\mathbb{Z})} := \sup_{t>0}\left\{t\,[\#\{k \in \mathbb{Z} : |x_k| > t\}]^{1/q}\right\} < \infty.$$

The space $l_{q,\omega}(\mathbb{N})$ is defined analogously.

Theorem 7.9. *For $q \in (1,\infty)$, $\{a_k(T)\} \in l_{q,\omega}(\mathbb{N})$ if and only if $\{\sigma_k\} \in l_{q,\omega}(\mathbb{Z})$, and*

$$\|\{\sigma_k\}\|_{l_{q,\omega}(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l_{q,\omega}(\mathbb{N})}.$$

Proof. Suppose $\{\sigma_k\} \in l_{q,\omega}(\mathbb{Z})$. From Corollary 3.3 and (7.23)

$$\begin{aligned}\|\{a_k(T)\}\|_{l_{q,\omega}(\mathbb{N})}^q &\preceq \sup_{t>0}\{t^q M(t)\} \\ &\leq \sup_{t>0}\{t^q N(t)\} \\ &\leq \sum_{n=1}^\infty t^q \#\{k : \sigma_k \geq nt/c\} \\ &\leq \sum_{n=1}^\infty \|\{\sigma_k\}\|_{l_{q,\omega}(\mathbb{Z})}^q \left(\frac{c}{n}\right)^q \preceq \|\{\sigma_k\}\|_{l_{q,\omega}(\mathbb{Z})}^q.\end{aligned}$$

Now suppose that $\{a_k(T)\} \in l_{q,\omega}(\mathbb{N})$. From Lemma 6.3,

$$\sup_{t>0}\left(t^q \# \{k \in \mathbb{Z} : \sigma_k > t\}\right) \preceq \sup_{t>0}\left(t^q\left(\#\left\{k \in \mathbb{N} : a_k(T) > \frac{t}{8}\right\}+1\right)\right).$$

Since

$$\#\left\{k \in \mathbb{N} : a_k(T) > \frac{t}{8}\right\} \geq \frac{M(t/8)}{2} - 2 \geq 1$$

for sufficiently small t, we conclude that

$$\sup_{t>0}\left(t^q \# \{k \in \mathbb{Z} : \sigma_k > t\}\right) \preceq \sup_{t>0}\left(t^q \#\left\{k \in \mathbb{N} : a_k(T) > \frac{t}{8}\right\}\right).$$

This implies that $\{\sigma_k\} \in l_{q,\omega}(\mathbb{Z})$ and $\|\{\sigma_k\}\|_{l_{q,\omega}(\mathbb{Z})} \preceq \|\{a_k(T)\}\|_{l_{q,\omega}(\mathbb{N})}$. The theorem is therefore proved. □

7.4 l_q and Weak-l_q Estimates, Cases $p = 1$ and ∞

We extend results from the previous section to cover the cases $p = 1$ and ∞.

The case $p = \infty$

The assumptions made on u, v here are that, for all $x \in (a,b)$,

$$u \in L_1(a,x), \tag{7.25}$$

and

$$v \in L_\infty(x,b). \tag{7.26}$$

We set $U(x) := \int_a^x |u(t)|dt$ and define $\xi_k \in \mathbb{R}^+$ by

$$U(\xi_k) = 2^k; \tag{7.27}$$

if $u \notin L_1(a,b)$, k may be any integer, but if $u \in L_1(a,b)$, k is constrained to satisfy $2^k \leq \|u\|_1$. For each admissible k we set

$$\sigma_k := \|uv\|_{\infty,Z_k}, \qquad Z_k = (\xi_k, \xi_{k+1}), \tag{7.28}$$

so that

$$2^k\|v\|_{\infty,Z_k} \leq \sigma_k \leq 2^{k+1}\|v\|_{\infty,Z_k}. \tag{7.29}$$

For non-admissible k we set $\sigma_k = 0$. The sequence $\{\sigma_k\}$ is the analogue of that introduced in the previous section. Then by a simple modification of the techniques used when $1 < p < \infty$ we obtain the following theorem.

Theorem 7.10. *Suppose that (7.25) and (7.25) are satisfied. Then*

(i) T is a map from $L_\infty(a,b)$ into $L_\infty(a,b)$ is bounded if and only if $\{\sigma_k\} \in l_\infty(\mathbb{Z})$, in which case

$$\|T\| \asymp \| \{\sigma_k\} \|_{l_\infty(\mathbb{Z})};$$

(ii) T is compact if and only if $\lim_{k\to\pm\infty} \sigma_k = 0$

(iii) If $\{\sigma_k\} \in l_1(\mathbb{Z})$,

$$\frac{1}{4}\int_a^b |u(t)|v_s(t)dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le 2\int_a^b |u(t)|v_s(t)dt,$$

where v_s is as in (6.20)

(iv) For $q \in (1,\infty)$, $\{a_k(T)\} \in l_q(\mathbb{N})$ if and only if $\{\sigma_k\} \in l_q(\mathbb{Z})$ and

$$\| \{\sigma_k\} \|_{l_q(\mathbb{Z})} \asymp \| \{a_k(T)\} \|_{l_q(\mathbb{N})};$$

(v) For $q \in (1,\infty)$, $\{a_k(T)\} \in l_{q,\omega}(\mathbb{N})$ if and only if $\{\sigma_k\} \in l_{q,\omega}(\mathbb{Z})$ and

$$\| \{\sigma_k\} \|_{l_{q,\omega}(\mathbb{Z})} \asymp \| \{a_k(T)\} \|_{l_{q,\omega}(\mathbb{N})}$$

The case $p = 1$

Here the assumptions (7.25) and (7.25) on u and v are replaced by

$$u \in L_\infty(a,x), \tag{7.30}$$

$$v \in L_1(x,b), \tag{7.31}$$

for all $x \in (a,b)$. On setting $a = -B$, $b = -A$, $\hat{f}(x) = f(-x)$, and similarly for u, v from the definition of T, we see that

$$T\hat{f}(x) = \hat{v}\,(x)\int_x^B \hat{u}\,(t)\hat{f}(t)dt, \quad A \le x \le B.$$

But this is the adjoint of the map $S : L_\infty(A,B) \to L_\infty(A,B)$ defined by

$$Sg(x) = \hat{u}\,(x)\int_A^x \hat{v}\,(t)g(t)dt, \quad A \le x \le B.$$

Hence, T and S have the same norms and their approximation numbers are equal if one, and hence both, are compact (see [41, Proposition II.2.5]). The results for $T : L_1(a,b) \to L_1(a,b)$ therefore follow from those proved for the $L_\infty(a,b)$ case on interchanging u and v. Before stating the results, we need some new terminology.

Let $\eta_k \in \mathbb{R}^+$ be defined by

$$V(x) := \int_x^b |v(t)|dt, \qquad V(\eta_k) = 2^k, \tag{7.32}$$

where $k \in \mathbb{Z}$ if $v \in L_1(a,b)$, but otherwise $2^k \leq \|v\|_1$. Set

$$\zeta_k := \|uv\|_{\infty,W_k}, \qquad W_k = (\eta_k, \eta_{k+1})$$

with $\zeta_k = 0$ if $v \in L_1(a,b)$ and $2^k > \|v\|_1$.

Theorem 7.11. *Suppose that (7.30) and (7.31) are satisfied. Then*

(i) T is a map from $L_1(a,b)$ into $L_1(a,b)$ is bounded if and only if $\{\zeta_k\} \in l_\infty(\mathbb{Z})$, in which case

$$\|T\| \asymp \|\{\zeta_k\}\|_{l_\infty(\mathbb{Z})};$$

(ii) T is compact if and only if $\lim_{k\to\pm\infty} \zeta_k = 0$

(iii) If $\{\zeta_k\} \in l_1(\mathbb{Z})$,

$$\frac{1}{4}\int_a^b u_s(t)|v(t)|dt \leq \liminf_{n\to\infty} n a_n(T) \leq \limsup_{n\to\infty} n a_n(T) \leq 2\int_a^b u_s(t)|v(t)|dt,$$

where u_s is as in (6.20)

(iv) For $q \in (1,\infty)$, $\{a_k(T)\} \in l_q(\mathbb{N})$ if and only if $\{\zeta_k\} \in l_q(\mathbb{Z})$ and

$$\|\{\zeta_k\}\|_{l_q(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l_q(\mathbb{N})};$$

(v) For $q \in (1,\infty)$, $\{a_k(T)\} \in l_{q,\omega}(\mathbb{N})$ if and only if $\{\zeta_k\} \in l_{q,\omega}(\mathbb{Z})$ and

$$\|\{\zeta_k\}\|_{l_{q,\omega}(\mathbb{Z})} \asymp \|\{a_k(T)\}\|_{l_{q,\omega}(\mathbb{N})}$$

See Remark 6.4 which demonstrates that v_s in the case $p = \infty$ (u_s when $p = 1$) is important.

Notes

Note 7.1. The special case of the remainder estimate of 7.1 when $p = 2$ was given in [52], the result for general p being provided in [83] and [84]. For the results in the later sections we refer to [58].

Chapter 8
A Non-Linear Integral System

Here we study the asymptotic behaviour of the eigenvalues of a non-linear integral system that arises from the problem of determining

$$\sup_{b \in T(B)} \|g\|_q ,$$

where B is the closed unit ball in $L_p(a,b)$ and $T : L_p(a,b) \to L_q(a,b)$ is the Hardy operator. This enables us to give the asymptotic behaviour of the approximation and Kolmogorov numbers of T when $q \le p$, and that of the Bernstein numbers of T when $p \le q$.

8.1 Upper and Lower Estimates for a Non-Linear Integral System

We consider the following non-linear system on $I = [a,b]$:

$$g(x) = (Tf)(x) \tag{8.1}$$

and

$$(f(x))_{(p)} = \lambda (T^*(g_{(q)}))(x), \tag{8.2}$$

where $(h(x))_{(r)} = |h(x)|^{r-2}h(x)$, T is the map $T_{a,(a,b),v,u}$ of Hardy type from (6.1) and T^* is the map defined by

$$(T^* f)(x) = u(x) \int_x^b v(y) f(y) dy.$$

Here we suppose that $p, q \in (1,\infty)$ and that u, v are positive functions on I such that $u \in L_{p'}(I)$ and $v \in L_q(I)$. The non-linear system (8.1) and (8.2) gives us the following non-linear equation:

$$(f(x))_{(p)} = \lambda T^*((Tf)_{(q)})(x). \tag{8.3}$$

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This is equivalent to its dual equation:

$$(s(x))_{(q')} = \lambda^* T((T^* s)_{(p')})(x), \tag{8.4}$$

and there is the following relation: For given f and λ satisfying (8.3) we have $s = (Tf)_{(q)}$ and $\lambda^* = \lambda_{(p')}$ satisfying (8.4), and for given s and λ^* satisfying (8.4) we have $f = (T^* s)_{(p')}$ and $\lambda = \lambda^*_{(q)}$ satisfying (8.3).

By a spectral triple will be meant a triple (g, f, λ) satisfying (8.1) and (8.2), where $\|f\|_p = 1$; (g, λ) will be referred to as a spectral pair; the function g corresponding to λ is called a spectral function and the number λ occurring in a spectral pair will be called a spectral number.

For the system (8.1) and (8.2) we denote by $SP(T, p, q)$ the set of all spectral triples; $sp(T, p, q)$ will stand for the set of all spectral numbers λ from $SP(T, p, q)$.

This non-linear system is related to the isoperimetric problem of determining

$$\sup_{g \in T(B)} \|g\|_q, \tag{8.5}$$

where $B := \{f \in L_p(I) : \|f\|_p \le 1\}$.

Moreover, this problem can be seen as a natural generalization of the p,q-Laplacian differential equation (see Sect. 3.2). For if u and v are identically equal to 1 on I, then (8.1) and (8.2) can be transformed into the p,q-Laplacian differential equation:

$$-\left(\left(w'\right)_{(p)}\right)' = \lambda (w)_{(q)}, \tag{8.6}$$

with the boundary condition

$$w(a) = 0. \tag{8.7}$$

If g, f and λ satisfy (8.1) and (8.2) then, the integrals being over I,

$$\begin{aligned}\int |g(x)|^q dx &= \int g(g)_{(q)} dx = \int Tf(x)(g)_{(q)} dx \\ &= \int f(x) T^*(g)_{(q)} dx = \lambda^{-1} \int f(x)(f)_{(p)} \\ &= \lambda^{-1} \int |f(x)|^p dx.\end{aligned}$$

From this it follows that $\lambda^{-1} = \|g\|_q^q / \|f\|_p^p$ and then for $(g_1, \lambda_1) \in SP(T, p, q)$ we have $\lambda_1^{-1/q} = \|g_1\|_q$.

Definition 8.1. Given any continuous function f on I we denote by $Z(f)$ the number of distinct zeros of f on $\overset{o}{I}$, and by $P(f)$ the number of sign changes of f on this interval. The set of all spectral triples (g, f, λ) with $Z(g) = n$ $(n \in \mathbb{N}_0)$ will be denoted by $SP_n(T, p, q)$, and $sp_n(T, p, q)$ will represent the set of all corresponding numbers λ. We set $\hat{\lambda}_n = \max sp_n(T, p, q)$ and $\check{\lambda}_n = \min sp_n(T, p, q)$.

We now give some results to prepare for the major theorems of this chapter and start by recalling the well known Borsuk Antipodal Theorem (see [109, p.14]).

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded, open, symmetric neighborhood of* 0, *and let T be a continuous map of $\partial\Omega$ into $\mathbb{R}^{m-1}$, with T odd on $\partial\Omega$ for all $x \in \partial\Omega$. Then there exists $y \in \partial\Omega$ such that $T(y) = 0$.*

Lemma 8.1. *Let $f \neq 0$ be a function on $[a,b]$ such that $Tf(a) = Tf(b) = 0$. Then $P(f) \geq 1$.*

Proof. This follows from the positivity of T and Rolle's theorem. □

Lemma 8.2. *Let $(g_i, f_i, \lambda_i) \in SP(T,p,q)$, $i = 1,2$, $1 < p,q < \infty$. Then for any $\varepsilon > 0$,*

$$P(Tf_1 - \varepsilon Tf_2) \leq P(Tf_1 - \varepsilon^{(p-1)/(q-1)}(\lambda_2/\lambda_1)^{1/(q-1)}Tf_2). \tag{8.8}$$

If the function $f_1 - \varepsilon f_2$ has a multiple zero and $P(Tf_1 - \varepsilon^{(p-1)/(q-1)}(\lambda_2/\lambda_1)^{q/(q-1)} Tf_2) < \infty$, then the inequality (8.8) is strict.

Proof. We use Lemma 8.1 and the fact that $\mathrm{sgn}(a-b) = \mathrm{sgn}((a)_{(p)} - (b)_{(p)})$.

$$\begin{aligned}
P(Tf_1 - \varepsilon Tf_2) &\leq Z(Tf_1 - \varepsilon Tf_2) \leq P(f_1 - \varepsilon f_2) \\
&\leq P((f_1)_{(p)} - \varepsilon^{p-1}(f_2)_{(p)}) \\
&\quad \text{(use (8.3) for } f_1 \text{ and } f_2\text{)}, \\
&\leq P(\lambda_1 T^*((g_1)_{(q)}) - \varepsilon^{p-1}\lambda_2 T^*((g_2)_{(q)})) \\
&\leq Z(\lambda_1 T^*((g_1)_{(q)}) - \varepsilon^{p-1}\lambda_2 T^*((g_2)_{(q)})) \\
&\leq P((g_1)_{(q)} - \varepsilon^{p-1}(\lambda_2/\lambda_1)(g_2)_{(q)}) \\
&\leq P(g_1 - \varepsilon^{(p-1)/(g-1)}(\lambda_2/\lambda_1)^{1/(q-1)}g_2) \\
&\leq P(Tf_1 - \varepsilon^{(p-1)/(q-1)}(\lambda_2/\lambda_1)^{1/(q-1)}Tf_2).
\end{aligned}$$

□

Theorem 8.2. *For all $n \in \mathbb{N}$, $SP_n(T,p,q) \neq \emptyset$.*

Proof. This uses ideas from [22] (see also [101]). For simplicity, suppose that I is the interval $[0,1]$. A key idea in the proof is the introduction of an iterative procedure used in [22].

Let $n \in \mathbb{N}$ and define

$$\mathscr{O}_n = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1} : \textstyle\sum_{i=1}^{n+1} |z_i| = 1 \right\}$$

and

$$f_0(x,z) = sgn(z_j) \text{ for } \textstyle\sum_{i=0}^{j-1} |z_i| < x < \sum_{i=1}^{j} |z_i|,\ j = 1, \ldots, n+1, \text{ with } z_0 = 0.$$

With $g_0(x,z) = Tf_0(x,z)$ we construct the iterative process

$$g_k(x,z) = Tf_k(x,z),\ f_{k+1}(x,z) = (\lambda_k(z)T^*(g_k(x,z))_{(q)})_{(p')},$$

where λ_k is a constant so chosen that

$$\|f_{k+1}\|_p = 1$$

and $1/p + 1/p' = 1$. Then, all integrals being over I,

$$\begin{aligned} 1 &= \int |f_k(x,z)|^p\,dx = \int f_k(f_k)_{(p)}dx = \int f_k\left([\lambda_{k-1}T^*((g_{k-1})_{(q)})]_{(p')}\right)_{(p)}dx \\ &= \int f_k\lambda_{k-1}T^*((g_{k-1})_{(q)})dx \\ &= \lambda_{k-1}\int T(f_k)(g_{k-1})_{(q)}dx \le \lambda_{k-1}\|g_k\|_q\|g_{k-1}\|_q^{q-1} \end{aligned}$$

and also

$$\begin{aligned} \|g_{k-1}\|_q^q &= \int |g_{k-1}(x,z)|^q\,dx = \int (g_{k-1})_{(q)}g_{k-1}dx \\ &= \int (g_{k-1})_{(q)}T(f_{k-1})dx = \int T^*((g_{k-1})_{(q)})f_{k-1}dx \\ &= \lambda_{k-1}^{-1}\int \lambda_{k-1}T^*((g_{k-1})_{(q)})f_{k-1}dx \\ &\le \lambda_{k-1}^{-1}\left(\int \left|(\lambda_{k-1}T^*((g_{k-1})_{(q)})_{(p')}\right|^{p'}dx\right)^{1/p'}\left(\int |f_{k-1}|^p\,dx\right)^{1/p} \\ &= \lambda_{k-1}^{-1}\left(\int \left|(\lambda_{k-1}T^*((g_{k-1})_{(q)})_{(p')}\right|^{p'}dx\right)^{1/p'} \\ &= \lambda_{k-1}^{-1}\left(\int |f_k|^p\,dx\right)^{1/p} = \lambda_{k-1}^{-1}. \end{aligned}$$

From these inequalities it follows that

$$\|g_{k-1}(\cdot,z)\|_q \le \lambda_{k-1}^{-1/q} \le \|g_k(\cdot,z)\|_q.$$

This shows that the sequences $\{g_k(\cdot,z)\}$ and $\{\lambda_k^{-1/q}(z)\}$ are monotonic increasing. Put $\lambda(z) = \lim_{k\to\infty}\lambda_k(z)$; then $\|g_k(\cdot,z)\|_q \to \lambda^{-1/q}(z)$.

As the sequence $\{f_k(\cdot,z)\}$ is bounded in $L_p(I)$, there is a subsequence $\{f_{k_i}(\cdot,z)\}$ that is weakly convergent, to $f(\cdot,z)$, say. Since T is compact, $g_{k_i}(\cdot,z) \to Tf(\cdot,z) := g(\cdot,z)$ and we also have $f(\cdot,z) = (\lambda(z)T^*(g(\cdot,z))_{(q)})_{(p')}$. It follows that for each $z \in \mathscr{O}_n$, the sequence $\{g_{k_i}(\cdot,z)\}$ converges to a spectral function.

Now set $z=(0,0,\dots,0,1)\in\mathscr{O}_n$. Then $f_0(\cdot,z)=1$, and as the operators T and T^* are positive, $g_k(\cdot,z)\geq 0$ for all k, so that $g(\cdot,z)\geq 0$. Thus $(g(\cdot,z),f(\cdot,z),\lambda(z))\in SP_0(T,p,q):SP_0(T,p,q)\neq\emptyset$.

Next we show that for all $n\in\mathbb{N}, SP_n(T,p,q)\neq\emptyset$. Given $n,k\in\mathbb{N}$, set

$$E_k^n=\{z\in\mathscr{O}_n : Z(g_k(\cdot,z))\leq n-1\}.$$

From the definition of T it follows that $g_k(\cdot,z)$ depends continuously on z; thus E_k^n is an open subset of $\mathscr{O}_n$ and $F_k^n:=\mathscr{O}_n\backslash E_k^n$ is a closed subset of $\mathscr{O}_n$. Let $0<t_1<\dots<t_n<1$ and put

$$F_k(\alpha)=(g_k(t_1,\alpha),\dots,g_k(t_n,\alpha)),\ \alpha\in\mathscr{O}_n.$$

Then F_k is a continuous, odd mapping from $\mathscr{O}_n$ to $\mathbb{R}^n$. By Borsuk's Theorem 8.1, there is a point $\alpha_k\in\mathscr{O}_n$ such that $F_k(\alpha_k)=0$; that is, $\alpha_k\in F_k^n$. From the definition of g_k and f_{k+1}, together with the positivity of T and T^*, we have

$$Z(g_{k+1})\leq P(f_{k+1})\leq Z(f_{k+1})\leq P(g_k)\leq Z(g_k),$$

so that $E_k^n\subset E_{k+1}^n$, which implies that $F_k^n\supset F_{k+1}^n$. Hence there exists $\widetilde{\alpha}\in\cap_{k\geq 1}F_k^n$, and as above we see that $g_k(\cdot,\widetilde{\alpha})$ converges, as $k\to\infty$, to a spectral function $g(\cdot,\widetilde{\alpha})\in SP_n(T,p,q)$. Thus $SP_n(T,p,q)\neq\emptyset$ and the proof is complete. □

We note that the previous theorem is true for much more general integral operators (i.e. integral operators with totally positive kernel, see [101]).

Next we obtain upper and lower estimates for eigenvalues of the non-linear system (8.1) and (8.2). To begin with, we establish an upper estimate for eigenvalues via the Kolmogorov numbers. We remind the reader that these numbers, for the map $T:L_p(I)\to L_q(I)$, are given by

$$d_{n+1}(T)=\inf_{X_n}\sup_{0<\|f\|_{p,I}\leq 1}\inf_{g\in X_n}\|Tf-g\|_{q,I}/\|f\|_{p,I},\qquad n\in\mathbb{N},$$

where the infimum is taken over all n-dimensional subspaces X_n of $L_q(I)$. The Makovoz lemma (see [22, Lemma 8.2.11]) will play a crucial role in our argument, and is as follows:

Lemma 8.3 (Makovoz). *Let $U_n\subset\{Tf:\|f\|_{p,I}\leq 1\}$ be a continuous and odd image of the sphere S^n in $\mathbb{R}^n$ endowed with the l_1 norm. Then*

$$d_{n+1}(T)\geq\inf\{\|x\|_{q,I}:x\in U_n\}.$$

Lemma 8.4. *If $n>1$, then $d_{n+1}(T)\geq\hat{\lambda}^{-1/q}$ where $\hat{\lambda}=\max\{\lambda\in\cup_{i=0}^n sp_i(p,q)\}$.*

Proof. Let us denote $\hat{\lambda}=\max\{\lambda\in\cup_{i=0}^n sp_i(p,q)\}$. The iteration process from the proof of Theorem 8.2 gives us, for each $k\in\mathbb{N}$ and $z\in\mathscr{O}_n$, a function $g_k(.,z)$. By the Makavoz lemma we have

$$d_{n+1}(T) \geq \max_{k\in\mathbb{N}} \min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_{q,I}. \tag{8.9}$$

Let us suppose that we have

$$\min_{z\in\mathcal{O}_n} \lim_{k\to\infty} \|g_k(.,z)\|_q = \max_{k\in\mathbb{N}} \min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_q. \tag{8.10}$$

Then from (8.9) and (8.10) it follows that

$$d_{n+1}(T) \geq \min_{z\in\mathcal{O}_n} \lim_{k\to\infty} \|g_k(.,z)\|_q \geq \hat{\lambda}^{-1/q},$$

since $\lim_{k\to\infty} g_k(.,z) \in SP(T,p,q)$. We have to prove (8.10). From the monotonicity of $\|g_k(.,z)\|_{q,I}$ we have

$$\max_{k\in\mathbb{N}} \min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_q = \lim_{k\to\infty} \min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_{q,}.$$

From max min $\leq$ min max it follows that

$$\begin{aligned} l := \lim_{k\to\infty} \min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_{q,} &= \max_{k\in\mathbb{N}} \min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_q \\ \leq \min_{z\in\mathcal{O}_n} \max_{k\in\mathbb{N}} \|g_k(.,z)\|_q &= \min_{z\in\mathcal{O}_n} \lim_{k\to\infty} \|g_k(.,z)\|_{q,} =: h. \end{aligned}$$

Denote $H_k(\varepsilon) = \{z \in \mathcal{O}_n; \|g_k(.,z)\|_q \leq h - \varepsilon\}$ where $0 < \varepsilon \leq h$.

Since the mapping $z \mapsto g_k(.,z)$ is continuous, $H_k(\varepsilon)$ is a closed subset of $\mathcal{O}_n$, and from the construction of the sequence g_k we see that $H_0(\varepsilon) \supset H_1(\varepsilon) \supset \dots$.

If $y_0 \in \cap_{k\in\mathbb{N}} H_k(\varepsilon) \neq \emptyset$ then $h = \min_{z\in\mathcal{O}_n} \lim_{k\to\infty} \|g_k(.,z)\|_q \leq \lim_{k\to\infty} \|g_k(.,y_0)\|_q \leq h - \varepsilon$, a contradiction. Then there exists $k_0 \in \mathbb{N}$ such that $H_k(\varepsilon) = \emptyset$ for $k \geq k_0$ and $\min_{z\in\mathcal{O}_n} \|g_k(.,z)\|_q \geq h - \varepsilon$ for $k \geq k_0$. Hence $h = l$ and (8.10) is proved. □

Next we recall the definition of the Bernstein numbers of $T\colon L_p(I) \to L_q(I)$:

$$b_n(T) := \sup_{X_n} \inf_{Tf\in X_n\setminus\{0\}} \|Tf\|_{q,I}/\|f\|_{p,I},$$

where the supremum is taken over all subspaces X_n of $T(L_p(I))$ with dimension n. Since u and v are functions for which $|\{x : u(x) = 0\}| = |\{x : v(x) = 0\}| = 0$, T is an injective function and the Bernstein numbers can be expressed as

$$b_n(T) = \sup_{X_n} \inf_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|T\left(\sum_{i=1}^n \alpha_i f_i\right)\|_{q,I}}{\|\sum_{i=1}^n \alpha_i f_i\|_{p,I}}, \tag{8.11}$$

where the supremum is taken over all n-dimensional subspaces $X_n = \text{span}\{f_1,\dots,f_n\} \subset L_p(I)$.

Now we use techniques from the proof of Theorem 8.2 to obtain an upper estimate for the Bernstein numbers.

Lemma 8.5. *If $n > 1$ then $b_{n+1}(T) \le \check{\lambda}^{-1/q}$, where $\check{\lambda} = \min(sp_n(p,q))$.*

Proof. Suppose there exists a linearly independent system of functions $\{f_1, \dots, f_{n+1}\}$ on I, such that:

$$\min_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\|T(\sum_{i=1}^{n+1} \alpha_i f_i)\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}} > \check{\lambda}^{-1/q}.$$

Let us define the n-dimensional 'sphere'

$$O_n = \left\{ T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right) : \|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I} = 1 \right\}.$$

Let $g_0(.) \in O_n$ and define a sequence of functions $h_k(.), g_k(.) = g_k(., g_0), k \in \mathbb{N}$, according to the following rule:

$$g_k(x) = Th_k(x), \qquad h_{k+1}(x) = (\lambda_k T^*(g_k(x))_{(q)})_{(p')},$$

where $\lambda_k > 0$ is a constant chosen so that $\|h_{k+1}\|_{p,I} = 1$.

We denote $O_n(k) = \{h_k(., h_0) : h_0(.) \in O_n\}$. As in the proof of Theorem 8.2 we have:

$\|g_k\|_{q,I}$ is a nondecreasing as $k \nearrow \infty$. For each $k \in \mathbb{N}$ there exists $g_k \in O_n(k)$ with n zeros inside I; $\lim_{k\to\infty} g_k(., g_0)$ is an eigenfunction and there exists $g_0(.)$ such that $\lim_{k\to\infty} g_k(., g_0)$ is an eigenfunction with n zeros. Moreover λ_k is monotonically decreasing as $k \nearrow \infty$.

Let $\overline{\alpha} \in \mathbb{R}^{n+1}$ be such that: $\overline{g_0}(.) = \left(\sum_{i=1}^{n+1} \overline{\alpha_i} f_i\right)$ is a function for which $\lim_{k\to\infty} \overline{g_k}(., g_0)$ is an eigenfunction with n zeros.

Then we have the following contradiction:

$$\min_{\alpha \in \mathbb{R}^n \setminus \{0\}} \frac{\|T\left(\sum_{i=1}^{n+1} \overline{\alpha_i} f_i\right)\|_{q,I}}{\|\sum_{i=1}^{n+1} \overline{\alpha_i} f_i\|_{p,I}} \le \|\overline{g_0}(.)\|_{q,I}$$
$$\le \lim_{k\to\infty} \|g_k(., \overline{g_0}(.))\|_{q,I} \le \check{\lambda}^{-1/q}.$$

□

Now we introduce functions $\mathscr{C}_0$ and $\mathscr{C}_+$ and study their properties together with those of the function $\mathscr{A}$ introduced in Definition 6.1.

Definition 8.2. Let $J = [c,d] \subset I$ and $x \in I$. Then

$$\mathscr{C}_{v,u,+}(J) := \mathscr{C}_+(J) := \sup\left\{ \frac{\|T_{c,J} f\|_{q,J}}{\|f\|_{p,J}} : f \in L_p(J) \setminus \{0\} \right\},$$

where $T_{c,J}$ is defined in (6.5);

$$\mathscr{C}_{v,u,0}(J) := \mathscr{C}_0(J) := \sup\left\{ \frac{\|Tf\|_{q,J}}{\|f\|_{p,J}} : f \in L_p(J)\backslash\{0\}, (Tf)(c) = (Tf)(d) = 0 \right\}.$$

From this definition we have:

Lemma 8.6. *Let I_1 and I_2 be intervals with $I_1 \subset I_2 \subset I$. Then*

$$\mathscr{C}_+(I_1) \le \mathscr{C}_+(I_2),$$

and

$$\mathscr{C}_0(I_1) \le \mathscr{C}_0(I_2), \quad \mathscr{C}_0(I_1) \le \mathscr{C}_+(I_1).$$

The quantities $\mathscr{C}_0(J)$ and $\mathscr{C}_+(J)$ are characterised in the next lemma.

Lemma 8.7. *Let $J = [c,d] \subset I$. Then for some $e \in \overset{0}{J}$,*

$$\mathscr{A}(J) = \|T_{e,J}|L_p(J) \to L_p(J)\| = \frac{\|g_1\|_{q,J}}{\|f_1\|_{p,J}} = \lambda_1^{-1/q},$$

where

$$(g_1, \lambda_1) \in SP(T,p,q) \text{ on } J \text{ and } g_1(e) = 0;$$

and

$$\mathscr{C}_+(J) = \frac{\|g_0\|_{q,J}}{\|f_0\|_{p,J}} = \lambda_0^{-1/q},$$

where

$$(g_0, \lambda_0) \in SP(T,p,q) \text{ on } J, \text{ and } g_0(c) = 0;$$

also

$$\mathscr{C}_0(J) = \|g_1\|_{q,J} = \lambda_1^{-q},$$

where

$$(g_1, \lambda_1) \in SP(T,p,q) \text{ on } J,\ g_1(c) = g_1(d) = 0.$$

Proof. Since T is a compact map from $L_p(J)$ to $L_q(J)$, there exist $h_0, h_1, h_2 \in L_p(J)$ and $x \in J$ such that:

(a) $\mathscr{A}(J) = \|T_{x,J} h_1\|_{q,J}, \|h_1\|_{p,J} = 1$
(b) $\mathscr{C}_+(J) = \|T_{c,J} h_0\|_{q,J}, \|h_0\|_{p,J} = 1$
(c) $\mathscr{C}_0(J) = \|T h_0\|_{q,J}, \|h_0\|_{p,J} = 1$

Put

$$G(f) = \|Tf\|_{q,J} / \|f\|_{p,J},\ f \neq 0.$$

Then $G'(f) = 0$ if, and only if, $Tf \in SP(T,p,q)$ on J. From (b) it follows that $G'(h_0) = 0$. By a simple modification of this argument, with the help of (a), the statement concerning A follows. The rest is proved in a similar manner. □

Next we give a monotonicity result. Note that the monotonicity of A was established in Lemma 6.7.

Lemma 8.8. *Let I_1, I_2 be intervals contained in I, with $I_1 \subset I_2$ and $|I_2 \backslash I_1| > 0$. Then*

(a) $\mathscr{C}_+(I_1) < \mathscr{C}_+(I_2)$
(b) $\mathscr{C}_0(I_1) < \mathscr{C}_0(I_2)$

Proof. First we prove (a) and consider the following cases:

(i) $I_1 = [c,d] \subset I_2 = [c,b]$, $d < b$
(ii) $I_1 = [c,d] \subset I_2 = [a,d]$, $a < c$
(iii) $I_1 = [c,d] \subset I_2 = [a,b]$, $a < c < d < b$

Clearly (a) will be established if we can handle these three cases. First suppose that (i) holds. Since T is compact, there exists $f_1 \geq 0$ such that

$$\mathscr{C}_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1} > 0.$$

Define f_2 on I_2 by $f_2(x) = f_1(x)$ if $x \in I_1$, $f_2(x) = 0$ if $x \in I_2 \backslash I_1$. Then $\|f_1\|_{p,I_1} = \|f_2\|_{p,I_2}$, $(Tf_1)(x) = (Tf_2)(x)$ $(x \in I_1)$, $(Tf_2)(x) > 0$ $(x \in I_2 \backslash I_1)$ and

$$\mathscr{C}_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1} < \|Tf_2\|_{q,I_1} / \|f_2\|_{p,I_1} \leq \mathscr{C}_+(I_2).$$

For case (ii), note that there exists $f_1 > 0$, with supp $f_1 \subset I_1$, such that

$$\mathscr{C}_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1} .$$

Since u is locally integrable, there exists $z \in \left(a, \frac{1}{2}(a+c)\right)$ such that $u(z) = \lim_{\varepsilon \to 0+} \int_z^{z+\varepsilon} u(x)dx$. Let $\delta > 0$ and define

$$f_2(x) = \delta \chi_{(z,z+\varepsilon)}(x) + f_1(x), \; x \in I_2.$$

Then for small $\delta > 0$ and $\varepsilon > 0$, there is a positive constant $\mathscr{C}_1$ such that

$$\|f_2\|_{p,I_2} \leq \mathscr{C}_1 \varepsilon^{1/p} \delta + \|f_1\|_{p,I_2} .$$

For Tf_2 we have, with $S(z) \approx \delta \varepsilon u(z)$,

$$(Tf_2)(x) \begin{cases} = 0, & a \leq x \leq z, \\ > 0, & z < x \leq z+\varepsilon, \\ = S(z)v(x), & z+\varepsilon < x \leq c, \\ = S(z)v(x) + (Tf_1)(x), & c < x \leq d. \end{cases}$$

From this it follows that for small positive δ and ε, there is a positive constant C_2 such that

$$\|Tf_2\|_{q,I_2} \geq \left\{ (S(z))^q \int_{z+\varepsilon}^{c} v^q(x)dx + \int_{c}^{d} |S(z)v(x) + (Tf_1)(x)|^q\, dx \right\}^{1/q}$$
$$\geq C_2\{(\delta\varepsilon)^q + \delta\varepsilon\} + \|Tf_1\|_{q,I_1}.$$

Hence for small positive δ and ε,

$$\frac{\|Tf_2\|_{q,I_2}}{\|f_2\|_{p,I_2}} \geq \frac{C_2\delta\varepsilon + \|Tf_1\|_{q,I_1}}{C_1\varepsilon^{1/p}\delta + \|f_1\|_{p,I_2}},$$

which implies that there exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for $\varepsilon = \varepsilon_1$ and $0 < \delta < \delta_1$,

$$\frac{\|Tf_2\|_{q,I_2}}{\|f_2\|_{p,I_2}} > \frac{\|Tf_1\|_{q,I_1}}{\|f_1\|_{p,I_1}}.$$

This gives the proof of (a) in case (ii). Case (iii) follows from (i) and (ii).

The proof of (b) can be accomplished by modification of this argument. □

Lemma 8.9. *The functions $\mathscr{C}_+([x,y])$, and $\mathscr{C}_0([x,y])$ are continuous in their dependence on x and y.*

Proof. Suppose that $\mathscr{C}_+([x,y])$ is not right-continuous as a function of the right-hand endpoint. Then there exist x and y, with $x < y$, and $t > 0$, such that

$$\mathscr{C}_+([x,y]) < t < \mathscr{C}_+([x,y+\varepsilon]) \text{ for all small enough } \varepsilon > 0. \tag{8.12}$$

Given each small enough $\varepsilon > 0$, there is a function f_ε such that

$$\mathscr{C}_+([x,y+\varepsilon]) = \frac{\|T_x f_\varepsilon\|_q}{\|f_\varepsilon\|_p},\ \text{supp } f_\varepsilon \subset [x,y+\varepsilon],\ \text{supp } T_x f_\varepsilon \subset [x,y+\varepsilon] \text{ and } \|f_\varepsilon\|_p = 1.$$

Since T_x is bounded, there exists $C > 0$ such that $\|T_x f_\varepsilon\|_q \leq C$. As T_x is compact, there is a sequence (ε_k) of positive numbers converging to zero and an element g of $L_q(I)$, with supp $g \subset \cap_k [x, y+\varepsilon_k] = [x,y]$, such that $T_x f_{\varepsilon_k} \to g$ in $L_q(I)$. From (8.12) we see that

$$\inf \|g - T_x f\|_{q,[x,y]} > 0, \tag{8.13}$$

where the infimum is taken over all f with supp $f \subset [x,y]$. However, since T_x has closed range, there exists $h \in L_p(I)$, with $\|h\|_p = 1$ and supp $h \subset [x,y]$, such that $Th = g$. This contradiction with (8.13) establishes the right-continuity of $\mathscr{C}_+$ in its dependence on the right-hand endpoint. Left continuity is proved in much the same way. Continuity of $\mathscr{C}_0$ can be proved by modification of the previous arguments. □

Of crucial importance is the next Lemma, which gives the values of functions $\mathscr{C}_+, \mathscr{C}_0$ and A when u and v are constant.

Lemma 8.10. *Let $u > 0$ and $v > 0$ be constant on the interval I. Then*

(i) $\mathscr{A}(I,u,v) = \mathscr{C}_{v,u,0}(I) = uv\,|I|^{1/p'+1/q}\mathscr{C}_{1,1,0}([0,1])$
(ii) $\mathscr{C}_{v,u,+}(I) = uv\,|I|^{1/p'+1/q}\mathscr{C}_{1,1,+}([0,1])$
(iii) $\mathscr{C}_{v,u,+}(I) = 2\mathscr{A}(I,u,v) = 2\mathscr{C}_{v,u,0}(I)$

Proof. For (ii) note that

$$\begin{aligned}
\mathscr{C}_{v,u,+}(I) &= \sup_{\text{supp } f\subset I} \frac{\|T_{v,u}f\|_{q,I}}{\|f\|_{p,I}} = \sup_{\text{supp } f\subset I} \frac{\|v\int_a^{\cdot} uf(t)dt\|_{q,I}}{\|f\|_{p,I}} \\
&= uv \sup_{\text{supp } f\subset I} \frac{\|\int_a^{\cdot} f(t)dt\|_{q,I}}{\|f\|_{p,I}} \\
&= uv\,|I|^{1/p'+1/q} \sup_{\text{supp } f\subset[0,1]} \frac{\|\int_0^{\cdot} f(t)dt\|_{q,[0,1]}}{\|f\|_{p,[0,1]}} \\
&= uv\,|I|^{1/p'+1/q}\mathscr{C}_{1,1,+}([0,1]).
\end{aligned}$$

In the same way we can prove (i). Finally, (iii) follows from (i) and (ii), together with Theorem 3.2, 3.5 and Lemma 8.7. □

From Theorem 5.8 we have

Lemma 8.11. *Let $f(t) = c(Sf)'(t)$, where $(Sf)(t) = c\sin_{pq}(\pi_{pq}t), (T_0f)(t) = c\sin_{pq}(\pi_{pq}t)$ and c is an arbitrary non-zero constant. Then*

$$\mathscr{A}([-1/2,1/2],1,1) = \frac{\|T_0 f\|_{q,[0,1]}}{\|f\|_{p,[0,1]}} = \frac{\|Sf\|_{q,[-1/2,1/2]}}{\|f\|_{p,[-1/2,1/2]}} = \frac{(p')^{1/q}q^{1/p'}(p'+q)^{1/p-1/q}}{2\pi_{pq}}$$

and

$$\mathscr{C}_{1,1,0}([0,1]) = \frac{\|Sf\|_{q,[0,1]}}{\|f\|_{p,[0,1]}} = \frac{(p')^{1/q}q^{1/p'}(p'+q)^{1/p-1/q}}{2\pi_{pq}}.$$

Note that $\mathscr{A}([0,1],1,1) = \mathscr{A}([-1/2,1/2],1,1)$ and the extremal functions for $\mathscr{A}([0,1],1,1)$ can be obtained by translation of the extremal function for $\mathscr{A}([-1/2,1/2],1,1)$. Now we establish the continuous dependence of $\mathscr{A}(I,u,v)$, $\mathscr{C}_{v,u,0}(I)$ and $\mathscr{C}_{v,u,+}(I)$ on u and v.

Lemma 8.12. *Let u_1,u_2 and v be positive weights on I with $u_1,u_2 \in L_{p'}(I)$ and $v \in L_q(I)$. Then*

(i) $|\mathscr{C}_{v,u_1,+}(I) - \mathscr{C}_{v,u_2,+}(I)| \le \|v\|_q\,\|u_1-u_2\|_{p'}$
(ii) $|\mathscr{C}_{v,u_1,0}(I) - \mathscr{C}_{v,u_2,0}(I)| \le 2\,\|v\|_q\,\|u_1-u_2\|_{p'}$

Proof. For (i), without loss of generality we may suppose that $\mathscr{C}_{v,u_1}(I) \geq \mathscr{C}_{v,u_2}(I)$. In what follows all the suprema are taken over all functions f such that $\operatorname{supp} f \subset I$ and $\|f\|_p \leq 1$. Then

$$\begin{aligned}
\mathscr{C}_{v,u_1,+}(I) &= \sup \left\| v(\cdot) \int_a^{\cdot} f(t) u_1(t) dt \right\|_q \\
&\leq \sup \left\{ \left\| v(\cdot) \int_a^{\cdot} f(t) \left| u_1(t) - u_2(t) \right| dt \right\|_q + \left\| v(\cdot) \int_a^{\cdot} f(t) u_2(t) dt \right\|_q \right\} \\
&\leq \sup \|v\|_q \|f\|_p \|u_1 - u_2\|_{p'} + \sup \left\| v(\cdot) \int_a^{\cdot} f(t) u_2(t) dt \right\|_q \\
&\leq \|v\|_q \|u_1 - u_2\|_{p'} + \mathscr{C}_{v,u2,+}(I).
\end{aligned}$$

Now we prove (ii). For $i = 0, 1$ set

$$\begin{aligned}
U_i &= \left\{ f : \int_a^b u_i(t) f(t) dt = 0, \|f\|_p = 1 \right\}, \\
V_i &= \left\{ f : \left| \int_a^b u_i(t) f(t) dt \right| \leq \|u_2 - u_1\|_{p'}, \|f\|_p = 1 \right\}.
\end{aligned}$$

Since

$$\left| \int_a^b u_1(t) f(t) dt \right| \leq \|u_2 - u_1\|_{p'} \|f\|_p + \left| \int_a^b u_2(t) f(t) dt \right|,$$

we have $U_2 \subset V_1$. Correspondingly, $U_1 \subset V_2$. Either $\mathscr{C}_{v,u_1,0}(I) \leq \mathscr{C}_{v,u_2,0}(I)$ or $\mathscr{C}_{v,u_1,0}(I) \geq \mathscr{C}_{v,u_2,0}(I)$. Suppose that the first case holds. Then

$$\begin{aligned}
\mathscr{C}_{v,u_2,0}(I) &= \sup_{f \in U_2} \left\| v(\cdot) \int_a^{\cdot} f(u_2 - u_1 + u_1) dt \right\|_q \\
&\leq \sup_{f \in U_2} \left\{ \|v\|_q \|u_2 - u_1\|_{p'} \|f\|_p + \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q \right\} \\
&\leq \|v\|_q \|u_2 - u_1\|_{p'} + \sup_{f \in U_1 \cup (V_1 \setminus U_1)} \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q \\
&\leq 2 \|v\|_q \|u_2 - u_1\|_{p'} + \sup_{f \in U_1} \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q .
\end{aligned}$$

Hence

$$\mathscr{C}_{v,u_2,0}(I) \leq 2 \|v\|_q \|u_2 - u_1\|_{p'} + \mathscr{C}_{v,u_1,0}(I).$$

The other case is handled similarly, and the proof of (ii) is complete. □

Lemma 8.13. *Let u, v_1 and v_2 be positive weights on I with $u \in L_{p'}(I)$ and $v_1, v_2 \in L_q(I)$. Then*

(i) $|\mathscr{C}_{v_2,u,+}(I) - \mathscr{C}_{v_1,u,+}(I)| \le \|v_1 - v_2\|_q \|u\|_{p'}$
(ii) $|\mathscr{C}_{v_2,u,0}(I) - \mathscr{C}_{v_1,u,0}(I)| \le \|v_1 - v_2\|_q \|u\|_{p'}$

Proof. The proof of (i) is just a simple modification of the previous proof. Let us prove (ii). The suprema in what follows are taken over all functions f such that $\operatorname{supp} f$, $\operatorname{supp} T_{v_1,u}f \subset I$ and $\|f\|_p \le 1$. Note that $\operatorname{supp} T_{v_1,u}f = \operatorname{supp} T_{v_2,u}f$. Then

$$\begin{aligned}
\mathscr{C}_{v_1,u,0}(I) &= \sup \left\| v_1(\cdot) \int_a^{\cdot} f(t)u(t)dt \right\|_q \\
&\le \sup \left\{ \left\| (v_1 - v_2) \int_a^{\cdot} f(t)u(t)dt \right\|_q + \left\| v_2 \int_a^{\cdot} f(t)u(t)dt \right\|_q \right\} \\
&\le \sup \left\{ \|v_1 - v_2\|_q \|f\|_p \|u\|_{p'} + \left\| v_2 \int_a^{\cdot} f(t)u(t)dt \right\|_q \right\} \\
&\le \|v_1 - v_2\|_q \|u\|_{p'} + \sup \left\| v_2 \int_a^{\cdot} f(t)u(t)dt \right\|_q \\
&\le \|v_1 - v_2\|_q \|u\|_{p'} + \mathscr{C}_{v_2,u,0}(I).
\end{aligned}$$

The rest is now clear. □

8.2 The Case $q \le p$

We introduce various techniques that will be used to establish the asymptotic theorem in the case $q \le p$. We suppose throughout this section that $u \in L_{p'}(I)$ and $v \in L_q(I)$: these assumptions are sufficient to ensure the compactness of T. We begin with an elementary lemma that is a simple consequence of Hölder's inequality.

Lemma 8.14. *Let $1 < q \le p < \infty$ and $n \in \mathbb{N}$. Then*

$$\sup_{\alpha \in \mathbb{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} = n^{1/q - 1/p},$$

the supremum being attained when $|\alpha_i| = 1$, $i = 1, \dots, n$; and

$$\inf_{\alpha \in \mathbb{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} = 1,$$

where the infimum is attained when $|\alpha_i| = 1$ for only one i and $\alpha_j = 0$ for each $j \ne i$.

Now we introduce functions that will be of crucial importance in our proofs.

Definition 8.3. Suppose that $0 < \varepsilon < \|T : L_p(I) \to L_q(I)\|$ and let $\mathscr{P}$ be the family of all partitions $P = \{a_1, \ldots, a_n\}$ of $[a,b]$, $a = a_1 < a_2 < \ldots < a_{n-1} < a_n = b$. Let

$$S(\varepsilon) : = \{n \in \mathbb{N} : \text{ for some } P = \{a_1, \ldots, a_n\} \in \mathscr{P}, \mathscr{C}_+([a_1,a_2]) \leq \varepsilon, \\ \mathscr{A}([a_2,a_3]) \leq \varepsilon, \ \ldots \ , \mathscr{A}([a_{n-1},a_n]) \leq \varepsilon\}$$

and define

$$B(\varepsilon) = \min S(\varepsilon) \text{ if } S(\varepsilon) \neq \emptyset, \ B(\varepsilon) = \infty \text{ otherwise.} \tag{8.14}$$

Monotonicity of B is clear:

Lemma 8.15. *If* $0 < \varepsilon_1 < \varepsilon_2 < \|T : L_p(I) \to L_q(I)\|$*, then* $B(\varepsilon_1) \geq B(\varepsilon_2)$.

We also have

Lemma 8.16. *Let* $0 < \varepsilon < \|T : L_p(I) \to L_q(I)\|$ *and suppose that* $B(\varepsilon) \geq 1$. *Put* $B(\varepsilon) = n$. *Then there is a partition* $P = \{a = a_1, a_2, \ldots, a_{B(\varepsilon)} = b\}$ *of* $[a,b]$ *such that* $\mathscr{C}_+([a_1,a_2]) = \varepsilon$, $\mathscr{A}([a_2,a_3]) = \varepsilon$, $\ldots$, $\mathscr{A}([a_{n-2},a_{n-1}]) = \varepsilon$, $\mathscr{A}([a_{n-1},a_n]) \leq \varepsilon$.

Proof. This follows from Lemmas 8.8 and 8.9, together with the techniques used in the proof of Lemma 6.11. □

Lemma 8.17. *Let* $T : L_p(I) \to L_q(I)$ *be compact. Then for all* $\varepsilon \in (0,C)$, $B(\varepsilon) < \infty$, *where* $C = \|T : L_p(I) \to L_q(I)\|$.

Proof. From the definition of compactness of T and a simple modification of the proof of Remark 6.2 and Lemma 6.11 the proof follows. □

Lemma 8.18. *Let* $n = B(\varepsilon_0)$ *for some* $\varepsilon_0 > 0$. *Then there exist* ε_1 *and* ε_2, $0 < \varepsilon_2 < \varepsilon_1 \leq \varepsilon_0$, *such that* $B(\varepsilon_2) = n+1$ *and* $B(\varepsilon_1) = n$; *and there is a partition* $\{a = a_1, a_2, \ldots, a_{B(\varepsilon_1)} = b\}$ *of* $[a,b]$ *such that the conclusion of Lemma 8.16 is satisfied with* $\mathscr{A}([a_{n-1},a_n]) = \varepsilon_1$.

Proof. We use the continuity of $\mathscr{C}_+([x,y])$ and $\mathscr{A}([x,y])$ as functions of the end-points x and y, together with the fact that $B(\varepsilon) < \infty$ for all $\varepsilon \in (0,C)$, where $C = \|T : L_p(I) \to L_q(I)\|$. Suppose that whenever $0 < \varepsilon \leq \varepsilon_0$, either $B(\varepsilon) > n+1$ or $B(\varepsilon) = n$. Put $\varepsilon_3 = \inf\{\varepsilon > 0 : \varepsilon \leq \varepsilon_0, B(\varepsilon) = n\}$. In view of the continuity properties of A and $\mathscr{C}_+$, if $\varepsilon_3 < \varepsilon \leq \varepsilon_0$, there is a sequence $a_1 = a, a_2, \ldots, a_n$ such that the conclusion of Lemma 8.16 is satisfied for the sequence with $\mathscr{C}_+([a_1,a_2]) = \varepsilon$, $\mathscr{A}([a_{i-1},a_i]) = \varepsilon$ if $2 \leq i \leq n-1$, and $\mathscr{A}([a_{n-1},a_n]) \leq \varepsilon$. Then there is a sequence $\{b_i\}_{i=1}^{n=B(\varepsilon_3)}$ such that $\mathscr{C}_+([a_1,a_2]) = \varepsilon$, $\mathscr{A}([b_{i-1},b_i]) = \varepsilon$ if $2 \leq i \leq n-1$, and $A([b_{n-1},b_n]) = \varepsilon$. Hence by the continuity of $\mathscr{C}_+$ and A there exists $\varepsilon < \varepsilon_3$ with $B(\varepsilon) = n+1$. The proof is complete. □

Our next objective is to make more precise the relationship between $B(\varepsilon)$ and ε. As before we suppose that $0 < u \in L_{p'}(I)$ and $0 < v \in L_q(I)$.

Lemma 8.19. *Let* $1 < q \le p < \infty$ *and* $r = 1/q + 1/p'$. *Then*

$$\lim_{\varepsilon \to 0+} \varepsilon B(\varepsilon)^r = A([0,1],1,1)\left(\int_I (uv)^{1/r} dt\right)^r.$$

Proof. Let $\beta > 0$. There are step functions u_β, v_β, with the same steps, such that $\|u_\beta - u\|_{p',I} \le \beta$, $\|v_\beta - v\|_{q,I} \le \beta$ and

$$\left|\int_I (uv)^{1/r} dt - \int_I (u_\beta v_\beta)^{1/r} dt\right| \le \beta.$$

Let $N(\beta)$ be the number of steps in the functions u_β, v_β and let $\varepsilon > 0$ be so chosen that $B(\varepsilon) \gg N(\beta)$. Let $\{J_i\}_{i=1}^{N(\beta)}$ be the set of all intervals on which u_β and v_β are constant, let $\{a_i\}_{i=1}^{N(\beta)}$ be the sequence from Lemma 8.16 and put $I_i = [a_{i-1}, a_i]$ for $i = 2, \ldots, B(\varepsilon)$. Plainly

$$I = \cup_{i=1}^{N(\beta)} J_i = \cup_{i=2}^{B(\varepsilon)} I_i.$$

Now define sets B, B_1 and B_2 by

$$B = \{1, \ldots, B(\varepsilon)\} = B_1 \cup B_2,$$

where

$$B_1 := \{i \in B : I_i \subset J_j \text{ for some } j, 1 \le j \le N(\beta)\},\ B_2 = B \backslash B_1.$$

Put

$$I_{B_1} = \cup_{i \in B_1} I_i,\ I_{B_2} = \cup_{i \in B_2} I_i.$$

Then for I_i $(i \in B_1 \backslash \{B(\varepsilon), 2\})$ we have, using Lemmas 8.10, 8.12 and 8.13,

$$\left|\mathscr{A}(I_i,u,v) - u_\beta v_\beta |I_i|^{1/p'+1/q} \mathscr{A}([0,1],1,1)\right| \le \|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i} + \|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i}.$$

We recall that for $0 < s < 1$, $\|.\|_s$ is a quasi-norm which satisfies the inequality $\|f+g\|_s^s \le \|f\|_s^s + \|g\|_s^s$ and we have also $\sum |f_i + g_i|^s \le \sum |f_i|^s + \sum |g_i|^s$.

Thus with the understanding that the summations are over all $i \in B_1 \backslash \{B(\varepsilon), 2\}$, together with help from this and the Hölder inequality,

$$\sum_i \left|\mathscr{A}(I_i,u,v) - u_\beta v_\beta |I_i|^{1/p'+1/q} \mathscr{A}([0,1],1,1)\right|^{1/r}$$
$$\le \sum_i \left(\|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i} + \|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i}\right)^{1/r}$$

$$\leq \sum_i (\|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i})^{1/r} + \sum_i (\|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i})^{1/r}$$
$$\leq \|u_\beta - u\|_{p',I}^{1/r} \|v\|_{q,I}^{1/r} + \|u\|_{p',I}^{1/r} \|v_\beta - v\|_{q,I}^{1/r},$$

and

$$\sum_i \left| \mathscr{A}(I_i,u,v) - u_\beta v_\beta |I_i|^{1/p'+1/q} \mathscr{A}([0,1],1,1) \right|^{1/r}$$
$$\geq \left| \sum_i \left| \mathscr{A}(I_i,u,v) \right|^{1/r} - \sum_i \left| u_\beta v_\beta |I_i|^{1/p'+1/q} \mathscr{A}([0,1],1,1) \right|^{1/r} \right|$$
$$\geq \left| \left\{ (\#B_1 - 1)\varepsilon^{1/r} \right\} - (\mathscr{A}([0,1],1,1))^{1/r} \left(\int_{I_{B_1 \setminus \{B(\varepsilon)\}}} (u_\beta v_\beta)^{1/r} \right) \right|.$$

Thus

$$\left| (\#B_1 - 1)\varepsilon^{1/r} - (\mathscr{A}([0,1],1,1))^{1/r} \left(\int_{I_{B_1 \setminus \{B(\varepsilon)\}}} (u_\beta v_\beta)^{1/r} \right) \right| \leq \beta^{1/r} (\|v\|_{q,I}^{1/r} + \|u\|_{p,I}^{1/r}).$$

When $\varepsilon \downarrow 0$, $I_{B_1 \setminus \{B(\varepsilon)}} \uparrow I$ and $\#B_1/\#B \uparrow 1$. Hence

$$\lim_{\varepsilon \to 0+} \left| \varepsilon(\#B)^r - \mathscr{A}([0,1],1,1) \left(\int_I (u_\beta v_\beta)^{1/r} \right)^r \right| \leq 2\beta (\|v\|_{q,I} + \|u\|_{p,I})$$

and the result follows. □

Next we establish a connection with the Kolmogorov widths for $T_{v,u}$.

Lemma 8.20. *Let $\varepsilon > 0$ be such that $B(\varepsilon) > 2$. Then*

$$a_{B(\varepsilon)}(T) \leq \varepsilon B(\varepsilon)^{1/q-1/p}.$$

Proof. Since T is compact, $B(\varepsilon) < \infty$. By Lemma 8.16, there are a sequence $\{a_i\}_{i=1}^{B(\varepsilon)}$ and intervals $I_i = [a_{i-1}, a_i]$ such that $\mathscr{C}_{v,u,+}(I_1) = \varepsilon$, $\mathscr{A}(I_i,u,v) = \varepsilon$ for $i = 2, \ldots, B(\varepsilon) - 1$ and $\mathscr{A}(I_{B(\varepsilon)},u,v) \leq \varepsilon$. For each i with $1 < i \leq B(\varepsilon) - 1$, denote by $c_i \in I_i$ a point such that

$$\mathscr{A}(I_i,u,v) = \sup_{f \in L^p(I_i)} \frac{\|T_{c_i,I_i} f\|_q}{\|f\|_p},$$

Put

$$P_{B(\varepsilon)} f(x) = \left[\sum_{i=2}^{B(\varepsilon)} (Tf)(c_i) \chi_{I_i}(x) \right] + 0 \cdot \chi_{I_1}(x);$$

this is a linear map $L_p \to L_q$ with rank $B(\varepsilon) - 1$.

We see that

$$\begin{aligned}
a_{B(\varepsilon)}(T) &\le \sup_{f\in L^p(I)} \frac{\|Tf - P_{B(\varepsilon)}f\|_{q,I}}{\|f\|_{p,I}} \\
&= \sup_{f\in L^p(I)} \frac{\left(\sum_{i=2}^{B(\varepsilon)} \|Tf(.) - Tf(c_i)\|_{q,I_i}^q + \|Tf(.)\|_{q,I_1}^q\right)^{1/q}}{\|f\|_{p,I}} \\
&\le \sup_{f\in L^p(I)} \frac{\left(\sum_{i=2}^{B(\varepsilon)} \|T_{c_i}f(.)\|_{q,I_i}^q + \|Tf(.)\|_{q,I_1}^q\right)^{1/q}}{\|f\|_{p,I}} \\
&\le \sup_{f\in L^p(I)} \frac{\varepsilon\left(\sum_{i=1}^{B(\varepsilon)} \|f\|_{p,I_i}^q\right)^{1/q}}{\|f\|_{p,I}} \\
&\le \sup_{f\in L^p(I)} \frac{\varepsilon[B(\varepsilon)]^{1/q-1/p}\left(\sum_{i=1}^{B(\varepsilon)} \|f\|_{p,I_i}^p\right)^{1/p}}{\|f\|_{p,I}} \\
&\le \varepsilon[B(\varepsilon)]^{1/q-1/p}.
\end{aligned}$$

□

Now we prove the reverse inequality with the Kolmogorov numbers.

Lemma 8.21. *Let* $1 < q \le p < \infty$. *Then*

$$\liminf_{n\to\infty} n d_n(T) \ge \mathscr{C}_{1,1,0}([0,1]) \left(\int_I |uv|^{1/r}\right)^r.$$

Proof. Let $n \in \mathbb{N}$ and define

$$\mathscr{O}_n = \left\{z = (z_1,\dots,z_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} |z_i| = 1\right\}.$$

For the sake of simplicity we suppose that $I = [a,b] = [0,1]$. We define

$$u_{n,z}(.) = \sum_{i=1}^{n+1} \chi_{I_i}(.)Tf_i(.)$$

where $z = (z_1, z_2, \dots, z_{n+1}) \in \mathscr{O}_n$, $I_j = [\sum_{i=0}^{j-1} |z_i|, \sum_{i=1}^{j} |z_i|]$, for $j = 1,\dots,n+1$, with $z_0 = 0$ and

$$\operatorname{supp} f_i = I_i, \quad f_i(t)\operatorname{sgn}(z_i) \ge 0 \text{ for all } t \in I,$$

$$\|f_i\|_{p,I_i} = 1, \qquad \frac{\|Tf_i\|_{q,I_i}}{\|f_i\|_{p,I_i}} = \mathscr{C}_{v,u,0}(I_i).$$

Then we can put $U_n = \{u_{n,z}(.) : z \in \mathscr{O}_n\}$ and have, with the aid of Lemmas 8.14 and 8.3,

$$d_n(T) \geq \inf\{\|u_{n,z}(.)\|_{q,I} : u_{n,z}(.) \in U_n\} n^{-1/p} = \inf\{\|u_{n,z}(.)\|_{q,I} : z \in \mathscr{O}_n\} n^{-1/p}.$$

Let $\beta > 0$. There are step functions u_β, v_β, with the same steps, such that $\left\|u_\beta - u\right\|_{p',I} \leq \beta$, $\left\|v_\beta - v\right\|_{q,I} \leq \beta$ and

$$\left| \int_I (uv)^{1/r} dt - \int_I (u_\beta v_\beta)^{1/r} dt \right| \leq \beta,$$

where $r = 1/q + 1/p'$.

Let $N(\beta)$ be the number of steps in the functions u_β, v_β; denote by $\{y_i\}_{i=1}^{N(\beta)}$ the set of points of discontinuity of u_β, v_β. We define $T_\beta f(.) = v_\beta(.) \int_a^. u_\beta(t) f(t) dt$ and

$$u_{n,z}^\beta(.) = \sum_{i=1}^{n+1} \chi_{I_i}(.) T_\beta f_i(.)$$

where $z = (z_1, z_2, \ldots, z_{n+1}) \in \mathscr{O}_n$, $I_j = [\sum_{i=0}^{j-1} |z_i|, \sum_{i=1}^{j} |z_i|]$, for $j = 1, \ldots, n+1$, with $z_0 = 0$ and

$$\operatorname{supp} f_i = I_j, \qquad f_i(t) \operatorname{sgn}(z_i) \geq 0 \text{ for all } t \in I,$$

$$\|f_i\|_{p,I_i} = 1, \qquad \frac{\|T_\beta f_i\|_{q,I_i}}{\|f_i\|_{p,I_i}} = \mathscr{C}_{v_\beta, u_\beta, 0}(I_i).$$

Putting $U_n^\beta = \{u_{n,z}^\beta(.) : z \in \mathscr{O}_n\}$ we have, again using Lemmas 8.3 and 8.14,

$$d_n(T_\beta) \geq \inf\{\|u_{n,z}^\beta(.)\|_{q,I}, u_{n,z}^\beta(.) \in U_n^\beta\} n^{-1/p} = \inf\{\|u_{n,z}^\beta(.)\|_{q,I}, z \in \mathscr{O}_n\} n^{-1/p}.$$

Now we modify the set U_n^β. Put

$$\tilde{u}_{n,z}^\beta(.) = \sum_i \chi_{J_i}(.) T_\beta f_i(.)$$

where $z = (z_1, z_2, \ldots, z_{n+1}) \in \mathscr{O}_n$, the J_i are intervals built from consecutive pairs of points from $\mathscr{P} := \{\sum_{i=1}^{j} |z_i|, j = 1, .., n+1\} \cup \{y_i, i = 1, .., N(\beta)\}$ and

$$\operatorname{supp} f_i = J_j, \qquad f_i(t) \operatorname{sgn}(z_i) \geq 0 \text{ for all } t \in I,$$

$$\|f_i\|_{p,J_i} = 1, \qquad \frac{\|T_\beta f_i\|_{q,J_i}}{\|f_i\|_{p,J_i}} = \mathscr{C}_{v_\beta, u_\beta, 0}(J_i).$$

Then with $\tilde{U}_n^\beta = \{\tilde{u}_{n,z}^\beta(.); z \in \mathscr{O}_n\}$ we have

$$d_n(T_\beta) \geq \inf\{\|u_{n,z}^\beta(.)\|_{q,I}, u_{n,z}^\beta(.) \in U_n^\beta\} n_\beta^{-1/p} \geq \inf\{\|\tilde{u}_{n,z}^\beta(.)\|_{q,I}, \tilde{u}_{n,z}^\beta(.) \in \tilde{U}_n^\beta\} n_\beta^{-1/p},$$

where $n \le n_\beta := \#\mathscr{P} \le n + N(\beta)$. It follows that

$$\|u_{n,z}(.)\|_q = (\sum_{i=1}^{n+1} (\mathscr{C}_{v,u,0}(I_i))^q)^{1/q} \ge (\sum_{j=1}^{n_\beta} (\mathscr{C}_{v,u,0}(J_j))^q)^{1/q},$$

and with the aid of Lemma 8.10:

$$\|\tilde{u}_{n,z}(.)\|_q = (\sum_{j=1}^{n_\beta} (\mathscr{C}_{v_\beta,u_\beta,0}(J_j))^q)^{1/q} = (\sum_{j=1}^{n_\beta} (u_\beta v_\beta \left|J_j\right|^{1/p'+1/q} \mathscr{C}_{1,1,0}([0,1]))^q)^{1/q}.$$

By Lemmas 8.12 and 8.13:

$$\begin{aligned}
&(\sum_{j=1} |\mathscr{C}_{v_\beta,u_\beta,0}(J_j) - \mathscr{C}_{v,u,0}(J_j)|^q)^{1/q} \le \\
&\quad \le (\sum_{j=1} |2 \left\|u_\beta - u\right\|_{p',J_i} \|v\|_{q,J_i} + \|u\|_{p',J_i} \left\|v_\beta - v\right\|_{q,J_i} |^q)^{1/q} \\
&\quad \le 2(\max_j \|u_\beta - u\|_{p',J_i})\|v\|_{q,I} + \|u\|_{p',I}\beta \\
&\quad \le \beta(2\|v\|_{q,I} + \|u\|_{p',I}).
\end{aligned}$$

From the definition of J_j and the Hölder inequality we have

$$\left[\int_I |u_\beta v_\beta|^{1/r}\right]^r = \left[\sum_{j=1}^{n_\beta} |u_\beta v_\beta|^{1/r} |J_j|\right]^r \le \left[\sum_{j=1}^{n_\beta} |u_\beta v_\beta|^q |J_j|^{1+q/p'}\right]^{1/q} n_\beta^{1/p'}.$$

By combining all these observations we have:

$$\begin{aligned}
\|u_n(z)\|_q &\ge \left(\sum_{j=1}^{n_\beta} (\mathscr{C}_{v,u,0}(J_j))^q\right)^{1/q} \\
&\ge \left(\sum_{j=1}^{n_\beta} \left(u_\beta v_\beta \left|J_j\right|^{1/p'+1/q} \mathscr{C}_{1,1,0}([0,1])\right)^q\right)^{1/q} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&\ge \mathscr{C}_{1,1,0}([0,1]) \left(\sum_{j=1}^{n_\beta} |u_\beta v_\beta|^{1/r} \left|J_j\right|\right)^r n_\beta^{-1/p'} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&= \mathscr{C}_{1,1,0}([0,1]) \left(\int_I |u_\beta v_\beta|^{1/r}\right)^r n_\beta^{-1/p'} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&\ge \mathscr{C}_{1,1,0}([0,1]) \left(\int_I (uv)^{1/r} dt\right)^r n_\beta^{-1/p'} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&\quad - \beta^r \mathscr{C}_{1,1,0}([0,1]) n_\beta^{-1/p'}.
\end{aligned}$$

Take small $\beta > 0$ and let $n \to \infty$: then $n_\beta/n \to 1$ and

$$\liminf_{n\to\infty} d_n(T)n \ge \mathscr{C}_{1,1,0}([0,1])\left(\int_I |uv|^{1/r}\right)^r - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) - \beta^r \mathscr{C}_{1,1,0}([0,1]) n_\beta^{-1/p'}.$$

The proof is completed by letting $\beta \to 0$. □

Theorem 8.3. *Suppose that* $0 < u \in L_{p'}(I)$, $0 < v \in L_q(I)$ *and* $1 < q \le p < \infty$. *Let* s_n *denote* $a_n(T)$ *or* $d_n(T)$. *Then*

$$\lim_{n\to\infty} ns_n = \mathscr{A}([0,1],1,1)\left(\int_I (uv)^{1/r}\right)^r,$$

where $r = 1/q + 1/p'$.

Proof. From the combination of Lemmas 8.19–8.21, the strict monotonicity of $B(\varepsilon)$ given by Lemma 8.18 and the fact that $a_n(T) \ge d_n(T)$, we have

$$A_{1,1}([0,1])\left(\int_I (uv)^{1/r}\right)^r = \lim_{\varepsilon\to 0} \varepsilon[B(\varepsilon)]^r = \lim_{\varepsilon\to 0} \varepsilon[B(\varepsilon)]^{1/q-1/p} B(\varepsilon)$$

$$\ge \limsup_{\varepsilon\to 0} a_{B(\varepsilon)} B(\varepsilon) = \limsup_{n\to\infty} a_n n \ge \limsup_{n\to\infty} nd_n$$

$$\ge \liminf_{n\to\infty} nd_n \ge A_{1,1}([0,1])\left(\int_I (uv)^{1/r}\right)^r.$$

The result follows. □

The following lemma give us a lower estimate for eigenvalues.

Lemma 8.22. *If* $n > 1$ *then* $a_n(T) \le \widehat{\lambda}^{-1/q}$, *where* $\widehat{\lambda} = \max(sp_n(p,q))$.

Proof. For the sake of simplicity we suppose that $|I| = 1$.

Let $(\widehat{g}, \widehat{f}, \widehat{\lambda}) \in SP_n(T,p,q)$. Denote by $\{a_i\}_{i=0}^n$ the set of zeros of $\widehat{g}$ (with $a_0 = a$) and by $\{b_i\}_{i=1}^{n+1}$ (with $b_{n+1} = b$) the set of zeros of $\widehat{f}$. Set $I_i = (b_i, b_{i+1})$ for $i = 1,\dots,n$ and $I_0 = (a_0, b_1)$, and define

$$T_n f(x) := \sum_{i=0}^n \chi_{I_i}(.)v(.)\int_a^{a_i} u(t)f(t)dt.$$

Then the rank of T_n is at most n.

We have (see Lemma 5.4) $d_n(T) \le a_n(T) \le \sup_{\|f\|_p \le 1} \|Tf - T_n f\|_q$.

Consider the extremal problem: determine

$$\sup_{\|f\|_p \le 1} \|Tf - T_n f\|_q. \tag{8.15}$$

This problem is equivalent to

$$\sup\{\|Tf\|_q : \|f\|_p \le 1, (Tf)(a_i) = 0 \text{ for } i = 0\dots n\}. \tag{8.16}$$

Since T and T_n are compact then there is a solution of this problem, that is, the supremum is attained. Let $\bar{f}$ be one such solution and denote $\bar{g} = T\bar{f}$. We can choose $\bar{f}$ such that $\bar{g}(t)\widehat{g}(t) \ge 0$, for all $t \in I$. We have $\|\bar{g}\|_{q,I} \ge \|\widehat{g}\|_{q,I}$.

Note that for any $f \in L^p(I)$ such that $Tf(a_i) = 0$ for every $i = 0,\dots,n$ we have $Tf(x) = T^+f(x)$ for each $x \in I$, where

$$T^+f(x) := \int_I K(x,t)f(t)dt = \sum_{i=0}^n \chi_{I_i}(.)v(.)\int_a^x u(t)f(t)dt$$

and

$$K(x,t) := \sum_{i=0}^n \chi_{I_i}(x)v(x)u(t)\chi_{(a_i,x)}\operatorname{sgn}(x-a_i).$$

Set $s(t) = |\hat{g}(t)|^q\hat{\lambda}^q$, where $\hat{\lambda} = \|\hat{g}\|_{q,I}$. Then, all integrals being over I, we have

$$\begin{aligned}
\left(\int |\bar{g}(t)|^q dt\right)^{1/q} &= \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{\bar{g}(t)}{\hat{g}(t)}\right|^q dt\right)^{1/q} \\
&\qquad \text{(use Jensen's inequality, noting that } \int s(t)dt = 1) \\
&\le \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{\bar{g}(t)}{\hat{g}(t)}\right|^p dt\right)^{1/p} \\
&= \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{T^+\bar{f}(t)}{\hat{g}(t)}\right|^p dt\right)^{1/p} \\
&= \hat{\lambda}^{-1/q}\left(\int s(t)\left|\frac{\int K(t,\tau)\bar{f}(\tau)d\tau}{\hat{g}(t)}\right|^p dt\right)^{1/p} \\
&= \hat{\lambda}^{-1/q}\left(\int s(t)\left|\int \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)}\frac{\bar{f}(\tau)}{\hat{f}(\tau)}d\tau\right|^p dt\right)^{1/p} \\
&\qquad \text{(use Jensen's inequality, noting that} \\
&\qquad \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)} \ge 0 \text{ and } \int \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)}d\tau = 1\Big) \\
&\le \hat{\lambda}^{-1/q}\left(\int s(t)\int \frac{K(t,\tau)\hat{f}(\tau)}{\hat{g}(t)}\left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p d\tau dt\right)^{1/p} \\
&= \hat{\lambda}^{-1/q}\left(\int \left|\frac{\bar{f}(\tau)}{\hat{f}(\tau)}\right|^p \hat{f}(\tau)\int \frac{K(t,\tau)s(t)}{\hat{g}(t)}dtd\tau\right)^{1/p}
\end{aligned}$$

$$= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \bar{f}(\tau) \int \frac{K(t,\tau)|\hat{g}(t)|^q}{\hat{g}(t)} \hat{\lambda} dt d\tau \right)^{1/p}$$

$$= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \bar{f}(\tau) \int K(t,\tau) \hat{g}_{(q)}(t) dt \hat{\lambda} d\tau \right)^{1/p}$$

$$\left(\text{use } \int K(t,\tau)\hat{g}_{(q)}(t) dt \hat{\lambda}^q = \hat{\lambda} T^*(\hat{g}_{(q)})(t) = \hat{f}_{(p)}(t) \right)$$

$$= \hat{\lambda}^{-1/q} \left(\int \left| \frac{\bar{f}(\tau)}{\hat{f}(\tau)} \right|^p \hat{f}(\tau) \hat{f}_{(p)}(\tau) d\tau \right)^{1/p}$$

$$(\text{ use } \hat{f}(t)\hat{f}_{(p)}(t) = |\hat{f}(t)|^p)$$

$$= \hat{\lambda}^{-1/q} \left(\int |\bar{f}(\tau)|^p d\tau \right)^{1/p} = \hat{\lambda}^{-1/q}.$$

From this it follows that $a_n(T) \leq \hat{\lambda}^{-1/q}$. □

Theorem 8.4. *If* $1 < q \leq p < \infty$, *then*

$$\lim_{n\to\infty} n\hat{\lambda}_n^{-1/q} = c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r$$

where $r = 1/p' + 1/q$, $\hat{\lambda}_n = \max(sp_n(p,q))$ *and*

$$c_{pq} = \frac{(p')^{1/q} q^{1/p'} (p'+q)^{1/p-1/q}}{2B(1/q, 1/p')}. \tag{8.17}$$

Proof. From Theorem 8.3 we have

$$\lim_{n\to\infty} n a_n(T) = \lim_{n\to\infty} n d_n(T) = \mathscr{A}([0,1],1,1) \left(\int_I |uv|^{1/r} dt \right)^r$$

and since $d_n(T) \leq a_n(T)$, $a_n(T) \searrow 0$ and $d_n(T) \searrow 0$, then from Lemma 8.22 it follows that

$$c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r \leq \liminf_{n\to\infty} n\hat{\lambda}_n^{-1/q},$$

and from Lemma 8.4 we have

$$\limsup_{n\to\infty} n\hat{\lambda}_n^{-1/q} \leq c_{pq} \left(\int_I |uv|^{1/r} dt \right)^r,$$

which finishes the proof. □

8.3 The Case $p \le q$

We start with various techniques that will be used to establish the main theorem. We suppose throughout this section that $0 < u \in L_{p'}(I)$ and $0 < v \in L_q(I)$. As in 8.2 we need an elementary lemma.

Lemma 8.23. *Let* $1 < p \le q < \infty$ *and* $n \in \mathbb{N}$. *Then*

$$\inf_{\alpha \in \mathbb{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} = n^{1/q-1/p},$$

and the infimum is attained when $|\alpha_i| = 1$, $i = 1, \ldots, n$.

We now introduce a function analogous to that given in Definition 8.3.

Definition 8.4. Suppose that $0 < \varepsilon < \|T : L_p(I) \to L_q(I)\|$ and let $\mathscr{P}$ be the family of all partitions $P = \{a_0, a_1, \ldots, a_n\}$ of $[a,b]$, $a = a_1 < a_2 < \ldots < a_{n-1} < a_n = b$. Let

$$S(\varepsilon) := \{n \in \mathbb{N} : \text{ for some } P \in \mathscr{P}, \mathscr{C}_0(a_{i-1}, a_i) \le \varepsilon \ (1 \le i \le n-1), \\ \mathscr{C}_+(a_{n-1}, a_n) \le \varepsilon\},$$

and define

$$B(\varepsilon) = \min S(\varepsilon) \text{ if } S(\varepsilon) \ne \emptyset, \ B(\varepsilon) = \infty \text{ otherwise.} \tag{8.18}$$

As an obvious consequence of this definition we have

Lemma 8.24. *If* $0 < \varepsilon_1 < \varepsilon_2 < \|T : L_p(I) \to L_q(I)\|$, *then* $B(\varepsilon_1) \ge B(\varepsilon_2)$.

We also have

Lemma 8.25. *Let* $0 < \varepsilon < \|T : L_p(I) \to L_q(I)\|$ *and suppose that* $B(\varepsilon) \ge 1$. *Then there is a partition* $P = \{a = a_0, a_1, \ldots, a_{B(\varepsilon)} = b\}$ *of* $[a,b]$ *such that* $\mathscr{C}_0([a_{i-1}, a_i]) = \varepsilon$ $(1 \le i \le B(\varepsilon) - 1)$, $\mathscr{C}_+([a_{B(\varepsilon)-1}, a_{B(\varepsilon)}]) \le \varepsilon$.

Proof. This follows directly from the monotonicity and continuity of $\mathscr{C}_0$ and $\mathscr{C}_+$, the proof being similar to that of Lemma 8.16. □

The next two lemmas can be proved in a way like that used for Lemmas 8.17 and 8.18.

Lemma 8.26. *For all* $\varepsilon \in (0, \|T : L_p(I) \to L_q(I)\|)$, $B(\varepsilon) < \infty$.

Lemma 8.27. *Let* $n = B(\varepsilon_0)$ *for some* $\varepsilon_0 > 0$. *Then there exist* ε_1 *and* ε_2, $0 < \varepsilon_2 < \varepsilon_1 \le \varepsilon_0$, *such that* $B(\varepsilon_2) = n + 1$ *and* $B(\varepsilon_1) = n$; *and there is a partition* $\{a = a_0, a_1, \ldots, a_{B(\varepsilon_1)} = b\}$ *of* $[a,b]$ *such that* $\mathscr{C}_0([a_{i-1}, a_i]) = \varepsilon_1$ *whenever* $1 \le i \le n-1$ *and* $\mathscr{C}_+([a_{n-1}, a_n]) = \varepsilon_1$.

Now we clarify the relation between $B(\varepsilon)$ and ε. As in the previous section we suppose that $0 < u \in L_{p'}(I)$ and $0 < v \in L_q(I)$.

Lemma 8.28. *Let* $1/r = 1/q + 1/p'$. *Then*

$$\lim_{\varepsilon \to 0+} \varepsilon B(\varepsilon)^{1/r} = \mathscr{C}_{1,1,0}([0,1]) \left(\int_I (uv)^r dt \right)^{1/r}.$$

Proof. Let $\beta > 0$. There are step functions u_β, v_β, with the same steps, such that $\left\| u_\beta - u \right\|_{p',I} \le \beta$, $\left\| v_\beta - v \right\|_{q,I} \le \beta$ and

$$\left| \int_I (uv)^r dt - \int_I (u_\beta v_\beta)^r dt \right| \le \beta.$$

Let $N(\beta)$ be the number of steps in the functions u_β, v_β and let $\varepsilon > 0$ be so chosen that $B(\varepsilon) \gg N(\beta)$. Let $\{J_i\}_{i=1}^{N(\beta)}$ be the set of all intervals on which u_β and v_β are constant, let $u_{\beta,i}, v_{\beta,i}$ be the constant values of u_β, v_β respectively on each J_i, let $\{a_i\}_{i=1}^{N(\beta)}$ be the sequence from Lemma 8.25 and put $I_i = [a_{i-1}, a_i]$ for $i = 1, \ldots, B(\varepsilon)$. Plainly

$$I = \cup_{i=1}^{N(\beta)} J_i = \cup_{i=1}^{B(\varepsilon)} I_i.$$

Now define sets B, B_1 and B_2 by

$$B = \{1, \ldots, B(\varepsilon)\} = B_1 \cup B_2,$$

where

$$B_1 := \{i \in B : I_i \subset J_j \text{ for some } j, 1 \le j \le N(\beta)\},\ B_2 = B \backslash B_1.$$

Put

$$I_{B_1} = \cup_{i \in B_1} I_i,\ I_{B_2} = \cup_{i \in B_2}.$$

Then for I_i $(i \in B_1 \backslash \{B(\varepsilon)\})$ we have, using Lemmas 8.10, 8.12 and 8.13,

$$\begin{aligned} \left| \mathscr{C}_{v,u,0}(I_i) - u_\beta v_\beta \left| I_i \right|^{1/p' + 1/q} \mathscr{C}_{1,1,0}([0,1]) \right| &\le 2 \left\| u_\beta - u \right\|_{p',I_i} \left\| v \right\|_{q,I_i} \\ &\quad + \left\| u \right\|_{p',I_i} \left\| v_\beta - v \right\|_{q,I_i}. \end{aligned}$$

Thus for $i \in B_1$, $i \ne B(\varepsilon)$, we have

$$\begin{aligned} \varepsilon^r &= \mathscr{C}_{v,u,0}(I_i)^r \\ &\ge \left\{ \mathscr{C}_{1,1,0}([0,1]) u_{\beta,i} v_{\beta,i} \left| I_i \right|^{1/p'+1/q} - 2 \left\| u_\beta - u \right\|_{p',I_i} \left\| v \right\|_{q,I_i} - \left\| u \right\|_{p',I_i} \left\| v_\beta - v \right\|_{q,I_i} \right\}^r \end{aligned}$$

and hence, with the understanding that the summations are over all $i \in B_1 \backslash \{B(\varepsilon)\}$,

$$\begin{aligned}
\{(\#B_1 - 1)\varepsilon^r\}^{1/r} &= \left(\sum_{i \in B_1 \backslash \{B(\varepsilon)\}} \mathscr{C}_{v,u,0}(I_i)^r \right)^{1/r} \\
&\geq \left\{ \sum \left(u_{\beta,i} v_{\beta,i} |I_i|^{1/p'+1/q} \right)^r \right\}^{1/r} \mathscr{C}_{1,1,0}([0,1]) \\
&\quad - 2 \left\{ \sum \left\| u_\beta - u \right\|_{p',I_i}^r \|v\|_{q,I_i}^r \right\}^{1/r} - \left\{ \sum \|u\|_{p',I_i}^r \left\| v_\beta - v \right\|_{q,I_i}^r \right\}^{1/r} \\
&\geq \mathscr{C}_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} - 2 \left\| u_\beta - u \right\|_{p',I} \|v\|_{q,I} \\
&\quad - \|u\|_{p',I} \left\| v_\beta - v \right\|_{q,I} \\
&\geq \mathscr{C}_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} - 3\beta(\|v\|_{q,I} + \|u\|_{p,I}).
\end{aligned}$$

Now we examine the upper bound for $\varepsilon^r \#B_1$. We have, as in the previous case,

$$\begin{aligned}
\{(\#B_1 - 1)\varepsilon^r\}^{1/r} &= \left(\sum_{i \in B_1 \backslash \{B(\varepsilon)\}} \mathscr{C}_{v,u,0}(I_i)^r \right)^{1/r} \\
&\leq \mathscr{C}_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} + 2\beta(\|v\|_{q,I} + \|u\|_{p,I}).
\end{aligned}$$

Thus

$$\left| (\#B_1 - 1)^{1/r}\varepsilon - \mathscr{C}_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} \right| \leq 3\beta(\|v\|_{q,I} + \|u\|_{p,I}).$$

When $\varepsilon \downarrow 0$, $I_{B_1 \backslash \{B(\varepsilon)\}} \uparrow I$ and $\#B_1/\#B \uparrow 1$. Hence

$$\lim_{\varepsilon \to 0+} \left| \varepsilon(\#B)^{1/r} - \mathscr{C}_{1,1,0}([0,1]) \left(\int_I (u_\beta v_\beta)^r \right)^{1/r} \right| \leq 3\beta(\|v\|_{q,I} + \|u\|_{p,I})$$

and the result follows. □

Next we obtain information about the Bernstein numbers b_n of $T_{v,u}$.

Lemma 8.29. *Let $\varepsilon > 0$ be such that $B(\varepsilon) > 2$. Then*

$$\varepsilon(B(\varepsilon) - 1)^{1/q - 1/p} \leq b_{B(\varepsilon)-2}.$$

Proof. Since T is compact, $B(\varepsilon) < \infty$. By Lemma 8.25, there are a sequence $\{a_i\}_{i=0}^{B(\varepsilon)}$ and intervals $I_i = [a_{i-1}, a_i]$ such that $\mathscr{C}_{v,u,0}(I_i) = \varepsilon$ for $i = 1, \dots, B(\varepsilon) - 1$ and $\mathscr{C}_{v,u,+}(I_{B(\varepsilon)}) \le \varepsilon$. For each i with $1 \le i \le B(\varepsilon) - 1$, denote by f_i a function such that supp f_i, supp $Tf_i \subset I_i$, $\|f_i\|_{p,I} = 1$ and $\mathscr{C}_{v,u,0}(I_i) = \|Tf_i\|_{q,I} / \|f_i\|_{p,I} = \varepsilon$. Put

$$X_{B(\varepsilon)} = \text{ span } \{f_i : i = 1, \dots, B(\varepsilon) - 1\};$$

this is a $(B(\varepsilon) - 1)$-dimensional subspace of $L_p(I)$. From (8.11) we see that

$$b_{B(\varepsilon)-2} \ge \inf_{\alpha \in \mathbb{R}^{B(\varepsilon)-1}} \frac{\left\| T(\sum_{i=1}^{B(\varepsilon)-1} \alpha_i f_i) \right\|_q}{\left\| \sum_{i=1}^{B(\varepsilon)-1} \alpha_i f_i \right\|_p}.$$

Now use Lemma 8.23. □

Lemma 8.30. *Let $\varepsilon > 0$ be such that $B(\varepsilon) > 2$. Then*

$$b_{B(\varepsilon)} \le (B(\varepsilon) - 2)^{1/q - 1/p} \varepsilon.$$

Proof. Suppose that there exists $\varepsilon > 0$ such that $B(\varepsilon) > 2$ and

$$(B(\varepsilon) - 2)^{1/q-1/p} \varepsilon < b_{B(\varepsilon)}.$$

Set $B(\varepsilon) = n$. Then there exists an $(n+1)$-dimensional subspace $X_{n+1} =$ span $\{f_1, \dots, f_{n+1}\}$ of $L_p(I)$ such that $T(X_{n+1})$ is an $(n+1)$-dimensional subspace of $L_q(I)$ and

$$b_n \ge \inf_{\alpha \in \mathbb{R}^{n+1}} \frac{\left\| T(\sum_{i=1}^{n+1} \alpha_i f_i) \right\|_q}{\left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_p} > (B(\varepsilon) - 2)^{1/q-1/p} \varepsilon.$$

Let

$$S_n := \left\{ \alpha \in \mathbb{R}^{n+1} : \left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_p = 1 \right\}$$

and put

$$u_0(\cdot, \alpha) = \sum_{i=1}^{n+1} \alpha_i f_i(\cdot)$$

for every $\alpha \in S_n$. For each $u_0(\cdot, \alpha)$ we construct an iterative process and a sequence $\{g_j(\cdot, \alpha)\}_{j \in \mathbb{N}}$ as follows:

$$g_j(\cdot, \alpha) = Tu_j(\cdot, \alpha),\ u_{j+1}(\cdot, \alpha) = \left(\lambda_j^q(\alpha) T^*((g_j(\cdot, \alpha))_{(q)}) \right)_{(p')},$$

where the $\lambda_j(\alpha)$ are chosen so that $\left\| u_{j+1}(\cdot, \alpha) \right\|_p = 1$.

Following arguments similar to those used in the proof of Theorem 8.2 we see that as j increases, $\|g_j(\cdot,\alpha)\|_q$ is monotone non-decreasing and $g_j(\cdot,\alpha)$ converges to a spectral function of (8.1) and (8.2). Moreover, if we let

$$g(\cdot,\alpha) := \lim_{j\to\infty} g_j(\cdot,\alpha) \text{ and } \lambda^{-1/q}(\alpha) := \lim_{j\to\infty} \|g_j(\cdot,\alpha)\|_q,$$

then $(g(\cdot,\alpha),\lambda(\alpha)) \in S(T,p,q)$ for every $\alpha \in S_n$. For each $l \in \mathbb{N}$ let

$$E_l^n := \{\alpha \in S_n : Z(g_l(\cdot,\alpha)) \le n-1\}.$$

From the definition of T we see that $g_j(\cdot,\alpha)$ depends continuously on α, and so, by the definition of S_n, it follows that E_l^n is an open subset of S_n for each $l \in \mathbb{N}$. Then $F_l^n := S_n \backslash E_l^n$ is a closed subset of S_n and $F_l^n \supset F_{l+1}^n$.

Take $\varepsilon > 0$ so that $B(\varepsilon) = n+1$ and with Lemma 8.27 in mind, let ε_1 be optimal in the sense that $B(\varepsilon_1) = n+1$ and $\varepsilon_1 := \inf\{\varepsilon > 0 : B(\varepsilon) = n\}$. Let $\{a_i\}_{i=1}^{n+1}$ be a sequence, forming a partition of I, such that

$$\mathscr{C}_0([a_{i-1},a_i]) = \varepsilon_1,\ (i = 2,\ldots,n), \quad \mathscr{C}_+([a_n,a_{n+1}]) = \varepsilon_1,$$

and put

$$F_l(\alpha) := (g_l(a_1,\alpha),\ldots,g_l(a_n,\alpha));$$

F_l is a continuous, odd mapping from S_n to $\mathbb{R}^n$, and by Borsuk's theorem, there exists $\alpha_l \in S_n$ such that $F_l(\alpha_l) = 0$, that is, $\alpha_l \in F_l^n$. There is a subsequence $\{\alpha_{l_k}\}_{k=1}^{\infty}$ of $\{\alpha_l\}_{l=1}^{\infty}$ with limit $\widetilde{\alpha} = \lim_{k\to\infty} \alpha_{l_k}$. Then $(g(\cdot,\widetilde{\alpha}),\lambda(\widetilde{\alpha})) \in S_n(T,p,q)$, and from the construction of $g_j(\cdot,\widetilde{\alpha})$ we have (see the proof of Theorem 8.2, Definition 8.2 and Lemma 8.7)

$$\min_{\alpha\in\mathbb{R}^{n+1}} \frac{\left\|T(\sum_{i=1}^{n+1}\alpha_i f_i)\right\|_q}{\left\|\sum_{i=1}^{n+1}\alpha_i f_i\right\|_p} \le \|g_j(\cdot,\widetilde{\alpha})\|_q \le \|g(\cdot,\widetilde{\alpha})\|_q = \lambda^{-1/q}(\widetilde{\alpha}).$$

Also

$$\mathscr{C}_0(I_i) = \frac{\|g(\cdot,\widetilde{\alpha})\|_{q,I_i}}{\|f(\cdot,\widetilde{\alpha})\|_{p,I_i}} = \varepsilon_1,\ I_i = [a_{i-1},a_i], i = 2,\ldots,n,$$

and

$$\mathscr{C}_+(I_{n+1}) = \frac{\|g(\cdot,\widetilde{\alpha})\|_{q,I_{n+1}}}{\|f(\cdot,\widetilde{\alpha})\|_{p,I_{n+1}}} = \varepsilon_1,\ I_{n+1} = [a_n,b].$$

Now let $G_{n+1} := \operatorname{span}\{\widetilde{f}_1,\ldots,\widetilde{f}_{n+1}\}$, where $\widetilde{f}_i(\cdot) := f_i(\cdot,\widetilde{\alpha})$. Then

$$\inf_{\alpha\in\mathbb{R}^{n+1}} \frac{\left\|T(\sum_{i=1}^{n+1}\alpha_i \widetilde{f}_i)\right\|_q}{\left\|\sum_{i=1}^{n+1}\alpha_i \widetilde{f}_i\right\|_p} = \|g(\cdot,\widetilde{\alpha})\|_q = \lambda^{-1/q}.$$

It can be seen that the infimum is attained when $\left\|\alpha_i \widetilde{f}_i\right\|_{p,I_i} = \left\|\alpha_j \widetilde{f}_j\right\|_{p,I_j}$. Then it follows that

$$\|g(\cdot,\widetilde{\alpha})\|_q = \varepsilon_1 B(\varepsilon_1)^{1/p-1/q},$$

and the proof is complete. □

Theorem 8.5. *Suppose that* $0 < u \in L_{p'}(I)$ *and* $0 < v \in L_q(I)$. *Then the Bernstein numbers of the compact map* $T : L_p(I) \to L_q(I)$ $(1 \le p \le q \le \infty)$ *satisfy*

$$\lim_{n\to\infty} nb_n = \mathscr{C}_{1,1,0}([0,1]) \left(\int_I (uv)^r\right)^{1/r},$$

where $1/r = 1/q + 1/p'$.

Proof. From the combination of Lemmas 8.28 and 8.29 and the strict monotonicity of $B(\varepsilon)$ given by Lemma 8.27 we have

$$\lim_{\varepsilon\to 0} \varepsilon[B(\varepsilon)]^{1/q+1/p'} = \lim_{\varepsilon\to 0} \varepsilon[B(\varepsilon)]^{1/q-1/p} B(\varepsilon) = \lim_{\varepsilon\to 0} b_{B(\varepsilon)} B(\varepsilon) = \lim_{n\to\infty} nb_n.$$

Together with Lemma 8.28 this completes the proof. □

Lemma 8.31. *Let* $1 < p \le q < \infty$ *and* $n > 1$. *Then* $b_n(T) \ge \check{\lambda}^{-1/q}$, *where* $\check{\lambda} = \min(sp_n(p,q))$.

Proof. We use the construction of Buslaev [21]. Take $(\check{g}, \check{f}, \check{\lambda})$ from $SP_n(T,p,q)$ and denote by $a = x_0 < x_1 < \ldots < x_i < \ldots < x_n < x_{n+1} = b$ the zeros of $\check{g}$. Set $I_i = (x_{i-1}, x_i)$ for $1 \le i \le n+1$, $f_i(.) = \check{f}(.)\chi_{I_i}(.)$ and $g_i(.) = \check{g}(.)\chi_{I_i}(.)$. Then $Tf_i = g_i(.)$ for $1 \le i \le n+1$.

Define $X_{n+1} = \text{span}\{f_1, \ldots f_{n+1}\}$. Since the supports of $\{f_i\}$ and $\{g_i\}$ are disjoint, then we have

$$b_n(T) \ge \inf_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right)\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}} = \inf_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|\sum_{i=1}^{n+1} \alpha_i g_i\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}}.$$

We shall study the extremal problem of finding

$$\inf_{\alpha\in\mathbb{R}^n\setminus\{0\}} \frac{\|\sum_{i=1}^{n+1} \alpha_i g_i\|_{q,I}}{\|\sum_{i=1}^{n+1} \alpha_i f_i\|_{p,I}}.$$

It is obvious that the extremal problem has a solution. Denote that solution by $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2, \ldots)$. Since $p \le q$, a short computation shows us that $\bar{\alpha}_i \ne 0$ for every i, moreover we can suppose that the $\bar{\alpha}_i$ alternate in sign. Label

$$\bar{\gamma} := \frac{\|\sum_{i=1}^{n+1} \bar{\alpha}_i g_i\|_{q,I}^q}{\|\sum_{i=1}^{n+1} \bar{\alpha}_i f_i\|_{p,I}^p};$$

then the solution of the extremal problem is given by $\bar{g} = \sum_{i=1}^{n+1} \bar{\alpha}_i g_i$, $\bar{f} = \sum_{i=1}^{n+1} \bar{\alpha}_i f_i$ where $\|\bar{f}\|_p = 1$.

Let us take the vector $\beta = (1, -1, \ldots)$. Define the functions $\tilde{g} = \sum_{i=1}^{n+1} \beta_i g_i$, $\tilde{f} = \sum_{i=1}^{n+1} \beta_i f_i$. Then

$$\lambda_n^{-1} := \frac{\| \sum_{i=1}^{n+1} \beta_i g_i \|_{q,I}^q}{\| \sum_{i=1}^{n+1} \beta_i f_i \|_{p,I}^p}.$$

It is obvious that $\bar{\gamma} \le \lambda_n^{-1}$. Suppose that $\bar{\gamma} < \lambda^{-1}$.

Since $\bar{\alpha}_i \ne 0$, $|\beta_i| = 1$ and $\bar{\gamma} < \lambda^{-1}$ then $0 < \varepsilon^* := \min_{1 \le i \le n+1} (\beta_i / \bar{\alpha}_i) < 1$. From Lemma 8.2 follows

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \le P(T(\tilde{f}) - \varepsilon^{*(p-1)/(q-1)} (\bar{\gamma}/\lambda_n^{-1})^{1/(q-1)} T(\bar{f})).$$

By repeated use of Lemma 8.2 with the help of $(\varepsilon^*)^{(p-1)/(q-1)} \le \varepsilon^* < 1$ and $\bar{\gamma}/\lambda^{-1} < 1$ we get

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \le P(T(\tilde{f})) = n.$$

On the other hand we have from Lemma 8.1 and the definition of ε^* that

$$P(T(\tilde{f}) - \varepsilon^* T(\bar{f})) \le P(\tilde{f} - \varepsilon^* \bar{f}) = P\left(\sum_{i=1}^{n+1} \beta_i f_i - \varepsilon^* \sum_{i=1}^{n+1} \bar{\alpha}_i f_i \right) \le n - 1,$$

which contradicts $\bar{\gamma} < \lambda^{-1}$. □

Theorem 8.6. *If* $1 < p \le q < \infty$ *then*

$$\lim_{n\to\infty} n \check{\lambda}_n^{-1/q} = c_{pq} \left(\int_I |uv|^r dt \right)^{1/r}$$

where $r = 1/p' + 1/q$, $\check{\lambda}_n = \min(sp_n(p,q))$ *and* c_{pq} *is as in (8.17).*

Proof. From [47] we have

$$\lim_{n\to\infty} n b_n(T) = c_{pq} \left(\int_I |uv|^r dt \right)^{1/r}$$

and since $b_n(T) \searrow 0$ then from Lemma 8.5 it follows that

$$c_{pq} \left(\int_I |uv|^r dt \right)^{1/r} \le \liminf_{n\to\infty} n \check{\lambda}_n^{-1/q}.$$

Moreover, from Lemma 8.31 we have

$$\limsup_{n\to\infty} n\check{\lambda}_n^{-1/q} \leq c_{pq}\left(\int_I |uv|^r dt\right)^{1/r}$$

which finishes the proof. □

8.4 The Case $p = q$

When $p = q$ the next Theorem follows from Theorems 8.4 to 8.6 (we can find this result in a sharper form in [7]).

Theorem 8.7. *If $p = q$ then*

$$\lim_{n\to\infty} n\lambda_n^{-1/q} = \lim_{n\to\infty} ns_n(T) = c_{pq}\left(\int_I |uv|^r dt\right)^{1/r}$$

where $r = 1/p' + 1/q$, c_{pq} is as in (8.17), λ_n is the single point in $sp_n(p,q)$ and $s_n(T)$ stands for $a_n(T), d_n(T)$ or $b_n(T)$.

Notes

Note 8.1. Compared with the case $p = q$ studied earlier, determination of the s-numbers of the (weighted) Hardy operator $T : L_p(I) \to L_q(I)$ when $p \neq q$ is a much more complex task and requires fresh ideas in conjunction with the techniques used in the simpler case. The results presented in this chapter are an amalgam of those given in [48–50].

Chapter 9
Hardy Operators on Variable Exponent Spaces

In this final chapter we introduce the spaces $L_{p(\cdot)}$ with variable exponent p and establish their basic properties. When I is a bounded interval (a,b) in $\mathbb{R}$ the Hardy operator $T_a : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)$ given by

$$T_a f(x) = \int_a^x f(t)dt$$

is studied: the asymptotic behaviour of its approximation, Bernstein, Gelfand and Kolmogorov numbers is determined. To conclude, a version of the $p(\cdot)$-Laplacian is presented and the existence established of a countable family of eigenfunctions and eigenvalues of the corresponding Dirichlet problem.

9.1 Spaces with Variable Exponent

Let Ω be a measurable subset of $\mathbb{R}^n$ with positive Lebesgue n-measure $|\Omega|$, let $\mathscr{M}(\Omega)$ be the family of all extended scalar-valued (real or complex) measurable functions on Ω and denote by $\mathscr{P}(\Omega)$ the subset of $\mathscr{M}(\Omega)$ consisting of all those functions p that map Ω into $(1,\infty)$ and satisfy

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1, \; p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty. \tag{9.1}$$

For every $f \in \mathscr{M}(\Omega)$ and $p \in \mathscr{P}(\Omega)$ define

$$\rho_p(f) = \int_\Omega |f(x)|^{p(x)}\,dx \tag{9.2}$$

and

$$\left\|f \mid L_{p(\cdot)}(\Omega)\right\| = \inf\left\{\lambda > 0 : \rho_p(f/\lambda) \le 1\right\}, \tag{9.3}$$

with the convention that $\inf \varnothing = \infty$. If no ambiguity is likely, we shall write $\|\cdot\|_p$ or $\|\cdot\|_{p,\Omega}$ instead of $\left\|\cdot \mid L_{p(\cdot)}(\Omega)\right\|$.

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Obviously $\rho_p(f) = \rho_p(-f) \geq 0$ for all $f \in \mathscr{M}(\Omega)$; and $\rho_p(f) = 0$ if and only if $f = 0$ a.e. Moreover, the function $f \longmapsto \rho_p(f)$ is convex: as this is less clear we indicate the proof. Let $f, g \in \mathscr{M}(\Omega)$. We use the inequality, valid for all $x \in \Omega$ and all $\lambda \in (0,1)$,

$$(\lambda + (1-\lambda)t)^{p(x)} \leq \lambda + (1-\lambda)t^{p(x)} \ (0 \leq t \leq 1)$$

which follows from the observation that

$$h(t) := (\lambda + (1-\lambda)t)^{p(x)} - \lambda - (1-\lambda)t^{p(x)}$$

is such that $h(1) = 0$ and $h'(t) \geq 0$. Thus at a point x such that $f(x) \neq 0$ and $|g(x)/f(x)| \leq 1$,

$$\begin{aligned} |\lambda f(x) + (1-\lambda)g(x)|^{p(x)} &= |f(x)|^{p(x)} \left| \lambda + (1-\lambda)\frac{g(x)}{f(x)} \right|^{p(x)} \\ &\leq |f(x)|^{p(x)} \left(\lambda + (1-\lambda) \left| \frac{g(x)}{f(x)} \right|^{p(x)} \right) \\ &= \lambda \, |f(x)|^{p(x)} + (1-\lambda) \, |g(x)|^{p(x)} . \end{aligned}$$

As we may limit ourselves to the consideration of such points x or corresponding points with f and g interchanged, it follows that

$$\rho_p(\lambda f + (1-\lambda)g) \leq \lambda \rho_p(f) + (1-\lambda)\rho_p(g),$$

which establishes the convexity of ρ_p. Hence ρ_p is a convex modular in the sense of Musielak [99].

It is clear that $\rho_p(f) \geq \rho_p(g)$ if $|f(x)| \geq |g(x)|$ for a.e. $x \in \Omega$; and that the map $\lambda \longmapsto \rho_p(f/\lambda)$ is continuous and decreasing on $[1,\infty)$ if $0 < \rho_p(f) < \infty$. Note that if $0 < \|f\|_p < \infty$, then

$$\rho_p\left(f/\|f\|_p\right) = 1. \tag{9.4}$$

To prove this, first observe that if $\gamma_k \downarrow \|f\|_p$, then by Fatou's lemma,

$$\rho_p\left(f/\|f\|_p\right) \leq \liminf_{k\to\infty} \rho_p(f/\gamma_k) \leq 1.$$

Moreover, if $0 < \lambda \leq \|f\|_p$, then

$$\rho_p(f/\lambda) \leq \left(\|f\|_p/\lambda\right)^{p+} \rho_p\left(f/\|f\|_p\right),$$

and so if $\rho_p(f/\|f\|_p) < 1$, there exists $\lambda \in (0, \|f\|_p)$ such that $\rho_p(f/\lambda) < 1$, which contradicts the definition of $\|f\|_p$.

Next, suppose that $0 < \|f\|_p \le 1$. Then since $\rho_p\left(f/\|f\|_p\right) = 1$, the convexity of ρ_p shows that

$$\rho_p(f) = \rho_p\left(\|f\|_p f/\|f\|_p\right) \le \|f\|_p \rho_p\left(f/\|f\|_p\right) = \|f\|_p. \tag{9.5}$$

It follows that

$$\rho_p(f) \le 1 \text{ if and only if } \|f\|_p \le 1. \tag{9.6}$$

However, this can be improved as follows (see [62]).

Proposition 9.1. *Let $p \in \mathscr{P}(\Omega)$. If $\|f\|_p < \infty$, then*

$$\min\left\{\|f\|_p^{p_-}, \|f\|_p^{p_+}\right\} \le \rho_p(f) \le \max\left\{\|f\|_p^{p_-}, \|f\|_p^{p_+}\right\}. \tag{9.7}$$

In particular, if (f_k) is a sequence in $\mathscr{M}(\Omega)$, then $\|f_k\|_p \to 0$ if and only if $\rho_p(f_k) \to 0$.

Proof. If $\|f\|_p = \lambda > 1$, then by (9.4),

$$\lambda^{-p_+}\rho_p(f) \le \rho_p(f/\lambda) = 1 \le \lambda^{-p_-}\rho_p(f),$$

and so $\|f\|_p^{p_-} \le \rho_p(f) \le \|f\|_p^{p_+}$. If $\|f\|_p < 1$ the analysis is similar. □

Our object now is to show that on an appropriate subset of $\mathscr{M}(\Omega)$, $\|\cdot\|_p$ is a norm. First we claim that for all $f_1, f_2 \in \mathscr{M}(\Omega)$, $\|f_1 + f_2\|_p \le \|f_1\|_p + \|f_2\|_p$. If the right-hand side is infinite there is nothing to prove; we therefore suppose that both $\|f_1\|_p$ and $\|f_2\|_p$ are finite, let $\lambda_i > \|f_i\|_p$ $(i = 1,2)$ and put $\lambda = \lambda_1 + \lambda_2$. By the convexity of ρ_p,

$$\begin{aligned}\rho_p\left(\frac{f_1+f_2}{\lambda}\right) &= \rho_p\left(\frac{f_1}{\lambda_1}\cdot\frac{\lambda_1}{\lambda} + \frac{f_2}{\lambda_2}\cdot\frac{\lambda_2}{\lambda}\right)\\ &\le \frac{\lambda_1}{\lambda}\rho_p\left(\frac{f_1}{\lambda_1}\right) + \frac{\lambda_2}{\lambda}\rho_p\left(\frac{f_2}{\lambda_2}\right) \le \frac{\lambda_1+\lambda_2}{\lambda} = 1.\end{aligned}$$

Hence $\|f_1 + f_2\|_p \le \lambda_1 + \lambda_2$. Since the λ_i may be chosen arbitrarily close to $\|f_i\|_p$ the result follows.

The claim that, for all $t \in \mathbb{R}$ and all f such that $\|f\|_p < \infty$, we have $\|tf\|_p = |t|\,\|f\|_p$ is obvious if $t = 0$. If $t \ne 0$, then

$$\begin{aligned}\|tf\|_p &= \inf\{\lambda > 0 : \rho_p(tf/\lambda) \le 1\} = |t| \inf\left\{\lambda/|t| > 0 : \rho_p\left(\frac{f}{\lambda/|t|}\right) \le 1\right\}\\ &= |t| \inf\{\beta > 0 : \rho_p(f/\beta) \le 1\} = |t|\,\|f\|_p,\end{aligned}$$

and the assertion is proved. Finally we show that $\|f\|_p = 0$ if and only if $f = 0$ a.e. Suppose that $\|f\|_p = 0$ and that $|f| > 0$ on a set of positive measure. Then there exists $\delta > 0$ such that $A := \{x \in \Omega : |f(x)| > \delta\}$ has positive measure. Hence given any $\lambda \in (0, \delta)$,

$$\rho_p(f/\lambda) = \int_\Omega |f(x)/\lambda|^{p(x)}\,dx \geq \int_A |f(x)/\lambda|^{p(x)}\,dx \geq \int_A |\delta/\lambda|^{p(x)}\,dx$$
$$\geq \int_A |\delta/\lambda|^{p_-}\,dx = |\delta/\lambda|^{p_-}\,|A| \geq 1$$

for all sufficiently small positive λ, say $\lambda \in (0, \lambda_0)$. Thus $\|f\|_p \geq \lambda_0 > 0$ and we have a contradiction. That $\|f\|_p = 0$ if $f = 0$ a.e. is obvious.

The properties of $\|\cdot\|_p$ that have been established make it clear that

$$L_{p(\cdot)}(\Omega) := \{f \in \mathscr{M}(\Omega) : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\} \tag{9.8}$$

is a linear space and that $\|\cdot\|_p$ is a norm on it. We formalise this in the next definition.

Definition 9.1. Let $p \in \mathscr{P}(\Omega)$. The space $L_{p(\cdot)}(\Omega)$ defined by (9.8) and endowed with the norm $\|\cdot\|_p$ is called a generalised Lebesgue space, or a Lebesgue space with variable exponent.

If p is a constant function, with $p(x) = p$ for all $x \in \Omega$, then $L_{p(\cdot)}(\Omega)$ coincides with the classical Lebesgue space $L_p(\Omega)$ and the norms on these spaces are equal. We remark that our restriction in the above definition to functions p that are bounded away from 1 and ∞ is made purely for ease of exposition, and refer to [81] for details of the theory without this limitation. Note also that $L_{p(\cdot)}(\Omega)$ spaces occur naturally in connection with various concrete questions, such as the study of variational problems with integrals having integrands satisfying non-standard growth conditions (see, for example, [124, 125]) and in the modelling of electrorheological fluids [112].

The spaces $L_{p(\cdot)}(\Omega)$ have various properties in common with their classical counterparts: we give some of the most interesting ones next, beginning with an extension of Hölder's inequality, for which we need the conjugate p' of $p \in \mathscr{P}(\Omega)$. As might be expected, this is defined by

$$p'(x) = p(x)/(p(x) - 1),\ x \in \Omega.$$

It is clear that $p' \in \mathscr{P}(\Omega)$.

Proposition 9.2. *Let $p \in \mathscr{P}(\Omega)$. Then for all $f \in L_{p(\cdot)}(\Omega)$ and all $g \in L_{p'(\cdot)}(\Omega)$,*

$$\int_\Omega |f(x)g(x)|\,dx \leq (1 + 1/p_- - 1/p_+)\,\|f\|_p\,\|g\|_{p'}\,.$$

Proof. We assume that $\|f\|_p\,\|g\|_{p'} \neq 0$, for otherwise the result is plain. Then for a.e. $x \in \Omega$, $x \in \Omega_0$, say, $1 < p(x) < \infty$, $|f(x)| < \infty$ and $|g(x)| < \infty$. In the inequality

$$ab \leq a^q/q + b^{q'}/q'\ (q' = q/(q-1),$$

take $a = |f(x)| / \|f\|_p$, $b = |g(x)| / \|g\|_{p'}$, $q = p(x)$ $(x \in \Omega_0)$, integrate over Ω_0 and use (9.4):

$$\int_{\Omega_0} \frac{|f(x)|}{\|f\|_p} \cdot \frac{|g(x)|}{\|g\|_{p'}} dx \leq \|1/p \mid L_\infty(\Omega_0)\| \rho_p \left(f / \|f\|_p\right) + \left\|1/p' \mid L_\infty(\Omega_0)\right\| \rho_{p'} \left(g / \|g\|_{p'}\right) \leq 1/p_- + 1 - 1/p_+.$$

The rest is clear. □

The next result provides a norm on $L_{p(\cdot)}(\Omega)$ equivalent to that given in (9.3).

Proposition 9.3. *For every $f \in \mathscr{M}(\Omega)$ put*

$$\|f\|_p' = \sup\left\{\int_\Omega f(x)g(x)dx : \rho_{p'}(g) \leq 1\right\}.$$

Then $L_{p(\cdot)}(\Omega) = \left\{f : \|f\|_p' < \infty\right\}$ and $\|\cdot\|_p'$ is a norm on $L_{p(\cdot)}(\Omega)$ equivalent to $\|\cdot\|_p$, with

$$\|\cdot\|_p \leq \|\cdot\|_p' \leq (1/p_- + 1 - 1/p_+) \|\cdot\|_p.$$

Proof. That $\|f\|_p' \leq (1/p_- + 1 - 1/p_+) \|f\|_p$ if $f \in L_{p(\cdot)}(\Omega)$ follows immediately from Proposition 9.2. For the rest we refer to [81], Theorem 2.3. □

We now recall the definition of a Banach function space.

Definition 9.2. A linear space $X \subset \mathscr{M}(\Omega)$ is called a Banach function space if there is a functional $\|\cdot\| : \mathscr{M}(\Omega) \to [0, +\infty]$ with the properties of a norm ($\|f\| = 0$ if and only if $f = 0$, $\|f + g\| \leq \|f\| + \|g\|$ and $\|\lambda f\| = |\lambda| \|f\|$ for all $f, g \in \mathscr{M}(\Omega)$ and all scalars λ) and such that:

(i) $f \in X$ if and only if $\|f\| < \infty$.
(ii) $\|f\| = \||f|\|$ for all $f \in \mathscr{M}(\Omega)$.
(iii) If $0 \leq f_k \uparrow f$, then $\|f_k\| \uparrow \|f\|$.
(iv) If $E \subset \Omega$ and $|E| < \infty$, then $\|\chi_E\| < \infty$.
(v) If $E \subset \Omega$ and $|E| < \infty$, then there is a constant $c(E)$ such that for all $f \in X$,

$$\int |f(x)| dx \leq c(E) \|f\|.$$

The classical Lebesgue and Sobolev spaces are Banach function spaces: see [6] and [42] for this and for other examples. In fact (see [53]), so are the spaces we have been considering here.

Proposition 9.4. *Let $p \in \mathscr{P}(\Omega)$. Then $\left(L_{p(\cdot)}(\Omega), \|\cdot\|_p\right)$ is a Banach function space.*

Proof. We check that conditions (i)–(v) are satisfied. Plainly (i), (ii) and (iv) hold, while (v) follows immediately from Hölder's inequality, Proposition 9.2. As for (iii), let (f_k) be a sequence in X with $0 \le f_k \uparrow f$. By monotone convergence, $\rho_p(f_k) \uparrow \rho_p(f)$ and $\left(\|f_k\|_p\right)$ is an increasing sequence bounded above by $\|f\|_p$. Suppose that $f \in L_{p(\cdot)}(\Omega)$; then given $\varepsilon \in \left(0, \|f\|_p\right)$, there exists $N \in \mathbb{N}$ such that

$$\rho_p\left(f_k/\left(\|f\|_p - \varepsilon\right)\right) > 1 \text{ if } k \ge N.$$

Hence $\|f_k\|_p > \|f\|_p - \varepsilon$ if $k \ge N$, and so $\|f_k\|_p \uparrow \|f\|_p$ as $k \uparrow \infty$. On the other hand, if $\|f\|_p = \infty$, then given any $k \in \mathbb{N}$, $\rho_p(f/k) > 1$ and so there exists $m(k) \in \mathbb{N}$ with $\rho_p\left(f_{m(k)}/k\right) > 1$; thus $\left\|f_{m(k)}\right\| > k$. It follows that $\|f_k\|_p \uparrow \infty$ and the proof is complete. □

Since it is known that every Banach function space X is a Banach space when endowed with the corresponding norm $\|\cdot\|$ (see, for example, [42], Theorem 3.1.3), we have immediately

Corollary 9.1. *Let $p \in \mathscr{P}(\Omega)$. Then $\left(L_{p(\cdot)}(\Omega), \|\cdot\|_p\right)$ is a Banach space.*

From Proposition 9.3 and (9.6) we see that if $p \in \mathscr{P}(\Omega)$, $g \in L_{p'(\cdot)}(\Omega)$ and G is defined on $L_{p(\cdot)}(\Omega)$ by

$$G(f) = \int_\Omega f(x)g(x)dx,\ f \in L_{p(\cdot)}(\Omega), \tag{9.9}$$

then G is a continuous linear functional on $L_{p(\cdot)}(\Omega)$ with

$$\|g\|_{p'(\cdot)} \le G \le (1 + 1/p_- - 1/p_+)\,\|g\|_{p'(\cdot)}\,.$$

More can be established: in fact (see [81], Theorem 2.6)

Theorem 9.1. *Let $p \in \mathscr{P}(\Omega)$. Then every continuous linear functional G on $L_{p(\cdot)}(\Omega)$ can be represented in the form (9.9) with a unique $g \in L_{p'(\cdot)}(\Omega)$.*

As an obvious consequence of this it follows that if $p \in \mathscr{P}(\Omega)$, then the dual of $L_{p(\cdot)}(\Omega)$ is (isometrically isomorphic to) $L_{p'(\cdot)}(\Omega)$ and $L_{p(\cdot)}(\Omega)$ is reflexive; in fact (see [62]), $L_{p(\cdot)}(\Omega)$ is even uniformly convex.

We also note that (see [81]) $L_{p(\cdot)}(\Omega)$ is separable, $C(\Omega) \cap L_{p(\cdot)}(\Omega)$ is dense in $L_{p(\cdot)}(\Omega)$ and, if Ω is open, $C_0^\infty(\Omega)$ is dense in $L_{p(\cdot)}(\Omega)$.

Now we turn to embeddings. When the underlying set Ω has finite measure $|\Omega|$, it is well known that the classical Lebesgue spaces $L_p(\Omega)$ are ordered: if $p < q$, then $L_q(\Omega) \hookrightarrow L_p(\Omega)$. As the following result (first proved in [81]) shows, the same is true for spaces with variable exponent.

Theorem 9.2. *Suppose that* $0 < |\Omega| < \infty$ *and that* $p, q \in \mathscr{P}(\Omega)$, *with* $p(x) \leq q(x)$ *for a.e.* $x \in \Omega$. *Then* $L_{q(\cdot)}(\Omega) \hookrightarrow L_{p(\cdot)}(\Omega)$ *and the corresponding embedding map id satisfies*

$$\|id\| \leq 1 + |\Omega|.$$

Proof. Let $f \in L_{q(\cdot)}(\Omega)$ be such that $\|f\|_{q(\cdot)} \leq 1$. Then by (9.6), $\rho_q(f) \leq 1$. Hence

$$\begin{aligned}\rho_p(f) &= \int_{|f(x)|>1} |f(x)|^{p(x)}\, dx + \int_{|f(x)|\leq 1} |f(x)|^{p(x)}\, dx \\ &\leq \int_{\Omega} |f(x)|^{q(x)}\, dx + |\Omega| \leq 1 + |\Omega|.\end{aligned}$$

Since ρ_p is convex,

$$\rho_p(f/(1+|\Omega|)) \leq (1+|\Omega|)^{-1}\rho_p(f) \leq 1,$$

and so, by (9.6) again, $\|f\|_p \leq 1 + |\Omega|$. The result follows. □

The condition $p(x) \leq q(x)$ for a.e. $x \in \Omega$ imposed in this theorem is necessary for the conclusion to be valid: see [81], Theorem 2.8. Note also that from Theorem 9.2 it can be shown (see [81]) that if (f_k) is a sequence in $L_p(\Omega)$ that converges to a function f in $L_p(\Omega)$, then there is a subsequence of (f_k) that converges pointwise a.e. in Ω to f.

Let $p, q \in \mathscr{P}(\Omega)$ be such that for some $\varepsilon \in (0,1)$,

$$p(x) \leq q(x) \leq p(x) + \varepsilon \text{ for all } x \in \Omega. \tag{9.10}$$

Our object now is to obtain upper and lower bounds for the norm of the embedding *id* of $L_{q(\cdot)}(\Omega)$ in $L_{p(\cdot)}(\Omega)$ that both approach 1 as $\varepsilon \to 0$. To this end we establish various preparatory lemmas.

Lemma 9.1. *Suppose that* $0 < |\Omega| < \infty$, *p and q satisfy (9.10) and that* $f \in \mathscr{M}(\Omega)$ *is such that* $\rho_q(f) \leq 1$. *Then*

$$\rho_p(f) \leq \varepsilon\, |\Omega| + \varepsilon^{-\varepsilon}.$$

Proof. Set

$$\begin{aligned}&\Omega_1 = \{x \in \Omega : |f(x)| < \varepsilon\},\ \Omega_2 = \{x \in \Omega : \varepsilon \leq |f(x)| \leq 1\}, \\ &\Omega_3 = \{x \in \Omega : 1 < |f(x)|\}.\end{aligned}$$

Then

$$\rho_p(f) = \sum_{j=1}^{3} \int_{\Omega_j} |f(x)|^{p(x)}\, dx = \sum_{j=1}^{3} A_j, \text{ say.}$$

Plainly

$$A_1 \leq \int_{\Omega_1} \varepsilon^{p(x)} dx \leq \int_{\Omega_1} \varepsilon dx \leq \varepsilon\,|\Omega|$$

and

$$A_3 \leq \int_{\Omega_3} |f(x)|^{q(x)}\,dx.$$

Moreover, on Ω_2 we have

$$\varepsilon^{\varepsilon} \leq \varepsilon^{q(x)-p(x)} \leq |f(x)|^{q(x)-p(x)} \leq 1,$$

so that

$$1 \leq |f(x)|^{p(x)-q(x)} \leq \varepsilon^{-\varepsilon}.$$

It follows that

$$A_2 = \int_{\Omega_2} |f(x)|^{q(x)}\,|f(x)|^{p(x)-q(x)}\,dx \leq \varepsilon^{-\varepsilon} \int_{\Omega_2} |f(x)|^{q(x)}\,dx.$$

Combination of these estimates gives

$$\begin{aligned}
\rho_p(f) &\leq \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon} \int_{\Omega_2} |f(x)|^{q(x)}\,dx + \int_{\Omega_3} |f(x)|^{q(x)}\,dx \\
&\leq \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon} \left(\int_{\Omega_2} |f(x)|^{q(x)}\,dx + \int_{\Omega_3} |f(x)|^{q(x)}\,dx \right) \\
&\leq \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon} \int_{\Omega} |f(x)|^{q(x)}\,dx \leq \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon}.
\end{aligned}$$

□

Lemma 9.2. *Suppose that* $0 < |\Omega| < \infty$ *and that p and q satisfy (9.10). Then*

$$\|id\| \leq \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon}.$$

Proof. Evidently $K := \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon} > 1$. Let f be such that $\rho_q(f) \leq 1$. Then by Lemma 9.1,

$$\rho_p(f/K) \leq K^{-1}\rho_p(f) \leq \left(\varepsilon\,|\Omega| + \varepsilon^{-\varepsilon}\right)/K = 1.$$

The result follows. □

Turning to estimates from below, we have

Lemma 9.3. *Suppose that p and q satisfy (9.10) and that* $1 \leq |\Omega| < \infty$. *Then* $\|id\| \geq 1$.

Proof. Define a function g by $g(x) = |\Omega|^{-1/q(x)}$ $(x \in \Omega)$. Then $\rho_q(g) = 1$. Since $|\Omega|^{-p(x)/q(x)} \geq |\Omega|^{-1}$ we have, for each $\lambda \in (0,1)$,

$$\rho_p(g/\lambda) = \int_\Omega \frac{|\Omega|^{-p(x)/q(x)}}{\lambda^{p(x)}} dx \geq \int_\Omega \frac{|\Omega|^{-1}}{\lambda^{p(x)}} dx \geq \int_\Omega \frac{|\Omega|^{-1}}{\lambda} dx = \lambda^{-1} > 1.$$

Thus $\|id\| \geq \lambda$ for each $\lambda \in (0,1)$, and so $\|id\| \geq 1$. □

Lemma 9.4. *Suppose that p and q satisfy (9.10) and that $0 < |\Omega| < 1$. Then $\|id\| \geq |\Omega|^\varepsilon$.*

Proof. Again we consider the function g given by $g(x) = |\Omega|^{-1/q(x)}$ $(x \in \Omega)$; $\rho_q(g) = 1$. Since

$$|\Omega|^{1-\frac{p(x)}{q(x)}} = |\Omega|^{\frac{q(x)-p(x)}{q(x)}} \geq |\Omega|^{\varepsilon/q(x)} \geq |\Omega|^\varepsilon,$$

it follows that

$$\rho_p(g) = \int_\Omega |\Omega|^{-p(x)/q(x)} dx = |\Omega|^{-1} \int_\Omega |\Omega|^{1-p(x)/q(x)} dx \geq |\Omega|^\varepsilon.$$

Hence, for each positive $\lambda < |\Omega|^\varepsilon$,

$$\begin{aligned} \rho_p(g/\lambda) &> \int_\Omega \left| \frac{g(x)}{|\Omega|^\varepsilon} \right|^{p(x)} dx = \int_\Omega \left| \frac{g(x)}{|\Omega|^{\varepsilon/p(x)}} \right|^{p(x)} dx \\ &= |\Omega|^{-\varepsilon} \int_\Omega |g(x)|^{p(x)} dx \geq |\Omega|^{-\varepsilon} |\Omega|^\varepsilon = 1. \end{aligned}$$

Thus $\|id\| \geq \lambda$ for each positive $\lambda < |\Omega|^\varepsilon$, which gives the result. □

The combination of these results leads immediately to the following theorem and corollary.

Theorem 9.3. *Suppose that $0 < |\Omega| < \infty$ and that p and q satisfy (9.10). Then the norm of the embedding id of $L_{q(\cdot)}(\Omega)$ in $L_{p(\cdot)}(\Omega)$ satisfies*

$$\min\left(1, |\Omega|^\varepsilon\right) \leq \|id\| \leq \varepsilon |\Omega| + \varepsilon^{-\varepsilon}.$$

Corollary 9.2. *Let $0 < |\Omega| < \infty$, let $p \in \mathscr{P}(\Omega)$ and suppose that for each $n \in \mathbb{N}$, $q_n \in \mathscr{P}(\Omega)$ and $\varepsilon_n > 0$, where $\lim_{n\to\infty} \varepsilon_n = 0$, and for all $n \in \mathbb{N}$ and all $x \in \Omega$,*

$$p(x) \leq q_n(x) \leq p(x) + \varepsilon_n.$$

Denote by id_n the natural embedding of $L_{q_n(\cdot)}(\Omega)$ in $L_{p(\cdot)}(\Omega)$. Then

$$\lim_{n\to\infty} \|id_n\| = 1.$$

Although the spaces with variable exponent have many properties in common with the classical Lebesgue spaces, important differences remain: for example, in general, elements of $L_{p(\cdot)}(\Omega)$ do not posses the property that is the natural analogue of p-mean continuity; Young's inequality for convolutions is false; and the Hardy–Littlewood maximal operator does not act boundedly from $L_{p(\cdot)}(\Omega)$ to itself. The imposition of conditions on p helps in this regard, the most common such condition being, when Ω is open and bounded, the following: there is a constant $C > 0$ such that

$$|p(x) - p(y)| \leq -C/\log|x-y| \text{ for all } x, y \in \Omega,\ 0 < |x-y| < 1/2. \tag{9.11}$$

If p satisfies this condition it is said to be log-Hölder continuous. By way of illustration of what can be acheived by this means we cite the result of Diening [35] concerning the Hardy–Littlewood maximal operator M, defined for each $f \in L_{1,loc}(\Omega)$, by

$$(Mf)(x) = \sup_{B \ni x} |B|^{-1} \int_{B\cap\Omega} |f(y)|\,dy,\ x \in \Omega,$$

where the supremum is taken over all balls B that contain x and for which $|B \cap \Omega| > 0$. This states that if Ω is open and bounded, and p satisfies (9.11), then there is a constant $C(\Omega, p)$ such that for all $f \in L_{p(\cdot)}(\Omega)$,

$$\|Mf\|_p \leq C(\Omega, p)\,\|f\|_p.$$

9.2 Hardy Operators

As before, let $I = [a,b]$ be a compact interval in the real line. Here we consider the Hardy operator T,

$$Tf(x) := \int_a^x f(t)dt,\ (x \in I),$$

and study its behaviour as a map between spaces with variable exponent. First we need to have conditions under which T is compact.

Lemma 9.5. *Let $1 < c < d < \infty$ and suppose that $p, q \in \mathscr{P}(I)$ are such that $p(x), q(x) \in (c,d)$ for all $x \in I$. Then T maps $L_{p(\cdot)}(I)$ compactly into $L_{q(\cdot)}(I)$.*

Proof. By Theorem 9.2, $L_{p(\cdot)}(I)$ and $L_d(I)$ are continuously embedded in $L_c(I)$ and $L_{q(\cdot)}(I)$, respectively. By Theorem 4.4, T maps $L_c(I)$ compactly into $L_d(I)$. The result now follows by composition of these maps. □

From now on, our concern is to determins the asymptotic behaviour of various s-numbers of T, and to do this we introduce functions of the kind used in the corresponding analysis for classical Lebesgue spaces.

Definition 9.3. Let $p,q \in \mathscr{P}(I)$, suppose that $J=(c,d)\subset I$ and let $\varepsilon>0$; set

$$p_J^- = \inf\{p(x) : x\in J\},\ p_J^+ = \sup\{p(x) : x\in J\}.$$

Then

$$A_{p(\cdot),q(\cdot)}(J) := \inf_{y\in J}\sup\left\{\left\|\int_y^{\cdot} f\right\|_{q,J} : \|f\|_{p,J}\le 1\right\},$$

$$B_{p(\cdot)}(J) := \inf_{y\in J}\sup\left\{\left\|\int_y^{\cdot} f\right\|_{p_J^+,J} : \|f\|_{p_J^-,J}\le 1\right\},$$

$$C_{p(\cdot),q(\cdot)}(J) := \sup\left\{\|Tf\|_{q,J} : \|f\|_{p,J}\le 1, (Tf)(c)=(Tf)(d)=0\right\}$$

and

$$D_{p(\cdot)}(J) := \sup\left\{\|Tf\|_{p_J^-,J} : \|f\|_{p_J^+,J}\le 1, (Tf)(c)=(Tf)(d)=0\right\}.$$

These quantities will sometimes be denoted by $A_{p(\cdot),q(\cdot)}(c,d)$, etc. Corresponding to these functions we define $N_{A_{p(\cdot),q(\cdot)}}(\varepsilon)$ to be the minimum of all those $n\in\mathbb{N}$ such that I can be written as $I=\bigcup_{j=1}^n I_j$, where each I_j is a closed sub-interval of I, $|I_i\cap I_j|=0$ $(i\ne j)$ and $A_{p(\cdot),q(\cdot)}(I_j)\le\varepsilon$ for every j. Quantities $N_{B_{p(\cdot)}}(\varepsilon), N_{C_{p(\cdot),q(\cdot)}}(\varepsilon)$ and $N_{D_{p(\cdot)}}(\varepsilon)$ are defined in an exactly similar way. For brevity we shall write $A_{p(\cdot)}(J)=A_{p(\cdot),p(\cdot)}(J)$ and $C_{p(\cdot)}(J)=C_{p(\cdot),p(\cdot)}(J)$, denoting these quantities by $A_p(J), C_p(J)$ respectively when p is a constant function. When p and q are constant functions we also write $A_{p,q}(J)=A_{p(\cdot),q(\cdot)}(J)$ and $C_{p,q}(J)=C_{p(\cdot),q(\cdot)}(J)$.

The techniques of Chap. 5 give the following result for the case of constant p and q.

Lemma 9.6. *Let $J=(c,d)\subset I$ and $p,q\in(1,\infty)$. Then*

$$\begin{aligned} A_{p,q}(J) = C_{p,q}(J) &= \frac{(p'+q)^{1/p-1/q}(p')^{1/q}q^{1/p'}}{2B(1/p',1/q)}\,|J|^{1/p'+1/q} \\ &:= \mathfrak{B}(p,q)\,|J|^{1/p'+1/q}. \end{aligned} \tag{9.12}$$

We now set about the task of establishing properties of the quantities introduced in Definition 9.3 similar to those known to hold when p and q are constant.

Lemma 9.7. *Let $p,q\in\mathscr{P}(I)$ and suppose that $(c,d)\subset I$. Then the functions $A_{p(\cdot),q(\cdot)}(c,t)$, $B_{p(\cdot)}(c,t)$, $C_{p(\cdot),q(\cdot)}(c,t)$ and $D_{p(\cdot)}(c,t)$ of the variable t are non-decreasing and continuous; $A_{p(\cdot),q(\cdot)}(t,d)$, $B_{p(\cdot)}(t,d)$, $C_{p(\cdot),q(\cdot)}(t,d)$ and $D_{p(\cdot)}(t,d)$ are non-increasing and continuous.*

Proof. We start with $A:=A_{p(\cdot),q(\cdot)}$ and first prove that $A(c,d)\le A(c,d+h)$ when $h\ge 0$. Clearly

$$A(c,d+h) = \inf_{y\in(c,d+h)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d+h)} : \|f\|_{p,(c,d+h)} \le 1 \right\}$$
$$= \min\{X,Y\},$$

where

$$X = \inf_{y\in(c,d)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d+h)} : \|f\|_{p,(c,d+h)} \le 1 \right\}$$

and

$$Y = \inf_{y\in(d,d+h)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d+h)} : \|f\|_{p,(c,d+h)} \le 1 \right\}.$$

Now

$$X \ge \inf_{y\in(c,d)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d)} : \|f\|_{p,(c,d)} \le 1 \right\} = A(c,d)$$

and

$$Y \ge \inf_{y\in(d,d+h)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d)} : \|f\|_{p,(c,d)} \le 1 \right\}$$
$$\ge \sup\left\{ \left\| \int_d^{\cdot} f(t)dt \right\|_{q,(c,d)} : \|f\|_{p,(c,d)} \le 1 \right\}$$
$$\ge \inf_{y\in(c,d)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d)} : \|f\|_{p,(c,d)} \le 1 \right\} = A(c,d),$$

which gives $A(c,d+h) \ge A(c,d)$. Next, we prove the continuity of A. By Hölder's inequality (Proposition 9.2) we have, for some $\alpha \ge 1$ (independent of f, x and y),

$$\left| \int_y^x f(t)dt \right| \le \alpha \|1\|_{p',(y,x)} \|f\|_{p,(y,x)},$$

and considering $\|1\|_{p'(\cdot),(y,x)}$ as a function of x we obtain

$$\left\| \|1\|_{p'(\cdot),(y,x)} \right\|_{q,(d,d+h)} \le \|1\|_{p',(c,d+h)} \|1\|_{q,(d,d+h)},$$

which gives

$$A(c,d) \le A(c,d+h)$$
$$= \inf_{y\in(c,d+h)} \sup\left\{ \left\| \int_y^{\cdot} f(t)dt \right\|_{q,(c,d+h)} : \|f\|_{p,(c,d+h)} \le 1 \right\}.$$

With the understanding that, unless otherwise specified, the suprema are taken over all f with $\|f\|_{p,(c,d+h)} \le 1$, we have

$$\begin{aligned}
A(c,d+h) &\le \inf_{y\in(c,d+h)} \sup\left\{\left\|\int_y^{\cdot} f(t)dt\right\|_{q,(c,d)} + \left\|\int_y^{\cdot} f(t)dt\right\|_{q,(d,d+h)}\right\} \\
&\le \inf_{y\in(c,d+h)} \sup\left\{\left\|\int_y^{\cdot} f(t)dt\right\|_{q,(c,d)} + \alpha\left\|\|1\|_{p',(y,x)}\,\|f\|_{p,(y,x)}\right\|_{q,(d,d+h)}\right\} \\
&\le \inf_{y\in(c,d+h)} \sup\left\{\left\|\int_y^{\cdot} f(t)dt\right\|_{q,(c,d)} + \alpha\left\|\|1\|_{p',(y,x)}\right\|_{q,(d,d+h)}\right\} \\
&\le \inf_{y\in(c,d+h)} \sup\left\|\int_y^{\cdot} f(t)dt\right\|_{q,(c,d)} + \alpha\,\|1\|_{p',(c,d+h)}\,\|1\|_{q,(d,d+h)} \\
&\le \inf_{y\in(c,d)} \sup\left\|\int_y^{\cdot} f(t)dt\right\|_{q,(c,d)} + \alpha\,\|1\|_{p',(c,d+h)}\,\|1\|_{q,(d,d+h)} \\
&\le \inf_{y\in(c,d)} \sup\left\{\left\|\int_y^{\cdot} f(t)dt\right\|_{q,(c,d)} : \|f\|_{p,(c,d)} \le 1\right\} \\
&\qquad + \alpha\,\|1\|_{p',(c,d+h)}\,\|1\|_{q,(d,d+h)} \\
&= A(c,d) + \alpha\,\|1\|_{p',(c,d+h)}\,\|1\|_{q,(d,d+h)}.
\end{aligned}$$

Since $q \in \mathscr{P}(I)$ we know that $\|1\|_{q,(d,d+h)} \to 0$ as $h \to 0$, and so $A(c,\cdot)$ is right-continuous. Left-continuity is proved in a corresponding manner and the continuity of $A(c,\cdot)$ follows. The arguments for B, C and D are similar. □

As an immediate consequence of this and Lemma 9.5 we have

Lemma 9.8. *Let $p \in \mathscr{P}(I)$. Then $T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)$ is compact and for all $\varepsilon > 0$ the quantities $N_{A_{p(\cdot)}}(\varepsilon)$, $N_{B_{p(\cdot)}}(\varepsilon)$, $N_{C_{p(\cdot),q(\cdot)}}(\varepsilon)$ and $N_{D_{p(\cdot)}}(\varepsilon)$ are finite.*

We also have

Lemma 9.9. *Let $p \in \mathscr{P}(I)$ and write $A = A_{p(\cdot)}$. Then given any $N \in \mathbb{N}$, there exists a unique $\varepsilon > 0$ such that $N_A(\varepsilon) = N$, and there is a covering of I by non-overlapping intervals I_A^i $(i = 1,...,N)$ such that $A(I_A^i) = \varepsilon$ for $i = 1,...,N$. The same holds when A is replaced by B, C, D.*

Proof. Existence follows from the continuity properties established in Lemma 9.7. For uniqueness, observe that given two such coverings of I, $\{I_A^i\}_{i=1}^N$ and $\{J_A^i\}_{i=1}^N$, there are m, j, k, l such that $I_A^m \subset J_A^j$ and $J_A^k \subset I_A^l$. Assuming that $A(I_A^i) = \varepsilon_1$ and $A(J_A^i) = \varepsilon_2$, we obtain $\varepsilon_1 \le \varepsilon_2 \le \varepsilon_1$ by the monotonicity of A. □

9.2.1 The Case when p is a Step-Function

Let $\{J_i\}_{i=1}^m$ be a covering of I by non-overlapping intervals and let p be the step-function defined by

$$p(x) = \sum_{i=1}^{m} \chi_{J_i}(x) p_i, \tag{9.13}$$

where $p_i \in (1,\infty)$ for each i. Again we shall write $A = A_{p(\cdot)}$ for brevity; B, C, D will have the analogous meaning.

Lemma 9.10. *Let p be the step-function given by (9.13). Then $T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)$ is compact and for small enough $\varepsilon > 0$,*

(i) $b_{N_C(\varepsilon)-m}(T) > \varepsilon$,
(ii) $a_{N_A(\varepsilon)+2m-1}(T) < \varepsilon$.

Proof. Let $\varepsilon > 0$. The compactness of T follows from Lemma 9.8, as does the finiteness of $N_A(\varepsilon)$ and $N_C(\varepsilon)$.

(i) By the continuity of $C(c,\cdot)$, there is a set $\{I_i : i = 1,...,N_C(\varepsilon)\}$ of non-overlapping intervals covering I and such that $C(I_i) = \varepsilon$ whenever $1 \le i < N_C(\varepsilon)$ and $C\left(I_{N_C(\varepsilon)}\right) \le \varepsilon$. Let $\eta \in (0,\varepsilon)$. Then corresponding to each i with $1 \le i < N_C(\varepsilon)$, there is a function f_i such that supp $f_i \subset I_i := (a_i, a_{i+1})$, $\|f_i\|_p = 1$, $\varepsilon - \eta < \|Tf_i\|_p \le \varepsilon$ and $(Tf)(a_i) = (Tf)(a_{i+1}) = 0$. By $\{I_{i_k}\}_{k=1}^M$ we denote the set of those intervals I_i, $1 \le i < N_C(\varepsilon)$, each of which is contained in one of the intervals J_l from the definition (9.13) of p. Then

$$N_C(\varepsilon) - m \le M \le N_C(\varepsilon).$$

Put

$$X_M = \left\{ f = \sum_{r=1}^{M} \alpha_{i_r} f_{i_r} : \alpha_{i_r} \in \mathbb{R} \right\};$$

this is an M-dimensional subspace of $L_{p(\cdot)}(I)$. Note that since p is constant on I_{i_r}, $p(x) = p_{i_r}$ on I_{i_r}. Choose $f \in X_M \backslash \{0\}$. With $\lambda_0 := \|Tf\|_{p(\cdot)}$ we have

$$\begin{aligned}
1 &\ge \int_I \left| \frac{Tf(x)}{\lambda_0} \right|^{p(x)} dx \ge \sum_{r=1}^{M} \int_{I_{i_r}} \left| \frac{Tf(x)}{\lambda_0} \right|^{p(x)} dx \\
&= \sum_{r=1}^{M} \left(\frac{1}{\lambda_0}\right)^{p_{i_r}} \int_{I_{i_r}} |Tf(x)|^{p_{i_r}} dx \ge \sum_{r=1}^{M} \left(\frac{\varepsilon - \eta}{\lambda_0}\right)^{p_{i_r}} \int_{I_{i_r}} |f(x)|^{p_{i_r}} dx \\
&= \sum_{r=1}^{M} \int_{I_{i_r}} \left| \frac{f(x)}{\lambda_0/(\varepsilon-\eta)} \right|^{p(x)} dx = \int_{\cup_{r=1}^{M} I_{i_r}} \left| \frac{f(x)}{\lambda_0/(\varepsilon-\eta)} \right|^{p(x)} dx \\
&= \int_I \left| \frac{f(x)}{\lambda_0/(\varepsilon-\eta)} \right|^{p(x)} dx.
\end{aligned}$$

Hence

$$\|f\|_{p,I} \le \|Tf\|_{p,I}/(\varepsilon-\eta),$$

and so $b_{N_C(\varepsilon)-m}(T) \ge b_M(T) \ge \varepsilon-\eta$.

(ii) This follows a pattern similar to that of (i). This time we let $\{I_i\}_{i=1}^{N_A(\varepsilon)}$ be a set of non-overlapping intervals covering I for which $A(I_i)=\varepsilon$ when $i=1,...,N_A(\varepsilon)-1$ and $A\left(I_{N_A(\varepsilon)}\right) \le \varepsilon$. By $\left\{I_i^+\right\}_{i=1}^{M}$ we denote the family of all non-empty intervals for which there exist j and k such that $I_i^+ = I_j \cap I_k$. Clearly $N_A(\varepsilon) \le M \le N_A(\varepsilon)+2(m-1)$. Let $\eta>0$. Then given any $i \in \{1,...,M\}$, there exists $y_i \in I_i^+$ such that

$$\sup\left\{\left\|\int_{y_i}^{\cdot} f\right\|_{p,I_i^+} : \|f\|_{p,I_i^+} = 1\right\} \le \varepsilon+\eta.$$

Define

$$P_\varepsilon(f) = \sum_{i=1}^{M}\left(\int_a^{y_i} f\right)\chi_{I_i^+}.$$

Plainly P_ε is a linear map from $L_{p(\cdot)}(I)$ to $L_{p(\cdot)}(I)$ with rank M. Let p_i be the constant value of p on I_i^+. Then for any $\lambda_0 \in (0,\infty)$ and $f \in L_{p(\cdot)}(I)$,

$$\begin{aligned}\int_I \left|\frac{(T-P_\varepsilon)f(x)}{\lambda_0}\right|^{p(x)} dx &= \sum_{i=1}^{M}\int_{I_i^+}\left|\frac{\int_{y_i}^{x} f}{\lambda_0}\right|^{p(x)} dx = \sum_{i=1}^{M}\lambda_0^{-p_i}\int_{I_i^+}\left|\int_{y_i}^{x} f\right|^{p_i} dx \\ &\le \sum_{i=1}^{M}\lambda_0^{-p_i}(\varepsilon+\eta)^{p_i}\int_{I_i^+}|f|^{p_i}\,dx = \int_I\left|\frac{f(x)}{\lambda_0/(\varepsilon+\eta)}\right|^{p(x)} dx.\end{aligned}$$

Now choose $\lambda_0 = (1-\eta)\,\|(T-P_\varepsilon)f\|_{p(\cdot),I}$. Then

$$1 < \int_I \left|\frac{(T-P_\varepsilon)f(x)}{\lambda_0}\right|^{p(x)} dx \le \int_I\left|\frac{f(x)}{\lambda_0/(\varepsilon+\eta)}\right|^{p(x)} dx,$$

from which we see that

$$\|f\|_{p,I} > (1-\eta)\,\|(T-P_\varepsilon)f\|_{p,I}/(\varepsilon+\eta),$$

so that

$$\frac{\varepsilon+\eta}{1-\eta} > \frac{\|(T-P_\varepsilon)f\|_{p,I}}{\|f\|_{p,I}}.$$

Now let $\eta \to 0$. □

Lemma 9.11. *Let p be the step-function given by (9.13). Then*

$$\lim_{\varepsilon\to 0}\varepsilon N(\varepsilon) = \frac{1}{2\pi}\int_I\left(p'(x)p(x)^{p(x)-1}\right)^{1/p(x)}\sin(\pi/p(x))dx,$$

where N stands for N_A, N_B, N_C or N_D.

Proof. Use the fact that p is a step function together with Lemmas 9.6 and 9.9. □

Finally, we have the main result when p is a step-function.

Theorem 9.4. *Let p be the step-function given by (9.13). Then for the compact map $T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)$ we have*

$$\lim ns_n(T) = \frac{1}{2\pi}\int_I \left(p'(x)p(x)^{p(x)-1}\right)^{1/p(x)} \sin(\pi/p(x))dx,$$

where $s_n(T)$ denotes the nth approximation, Gelfand, Kolmogorov or Bernstein number of T.

Proof. From Lemma 9.10 together with Theorems 5.4 and 5.5 it follows that

$$\varepsilon N_A(\varepsilon) \geq a_{N_A(\varepsilon)+2m-1}(T)N_A(\varepsilon) \geq b_{N_A(\varepsilon)+2m-1}(T)N_A(\varepsilon)$$

and

$$\varepsilon N_C(\varepsilon) \leq b_{N_C(\varepsilon)-m}(T)N_C(\varepsilon).$$

Use of Lemma 9.11 now gives the result for the approximation and Bernstein numbers. The rest follows from the inequalities of Theorems 5.4 and 5.5. □

9.2.2 The Case when p is Strongly Log-Hölder-Continuous

To obtain a result in this case similar to that of Theorem 9.4 the idea is to approximate p by step-functions. Corollary 9.2 enables estimates to be made of the changes in various norms occurring when p is replaced by an approximating function. First we give some technical lemmas.

Lemma 9.12. *Let $\delta > 0$, let $J \subset I$ be an interval and suppose that $p, q \in \mathscr{P}(J)$ are such that*

$$p(x) \leq q(x) \leq p(x) + \delta \text{ for all } x \in J.$$

Then

$$\left(\delta\,|J| + \delta^{-\delta}\right)^{-2} A_{p(\cdot)+\delta,p(\cdot)}(J) \leq A_{q(\cdot)}(J) \leq \left(\delta\,|J| + \delta^{-\delta}\right)^{2} A_{p(\cdot),p(\cdot)+\delta}(J).$$

Proof. Set

$$B_1 = \{f : \|f\|_q \leq 1\}, B_2 = \{f : \|f\|_p \leq \delta\,|J| + \delta^{-\delta}\},$$

where the norms are with respect to the interval J. By Theorem 9.3, $\|f\|_p \leq \left(\delta\,|J| + \delta^{-\delta}\right)\|f\|_q$, which gives $B_1 \subset B_2$ and

$$
\begin{aligned}
A_{q(\cdot)}(J) &= \inf_{y\in J}\sup\left\{\left\|\int_y^{\cdot} f\right\|_q : \|f\|_q \le 1\right\} = \inf_{y\in J}\sup\left\{\left\|\int_y^{\cdot} f\right\|_q : f\in B_1\right\} \\
&\le \inf_{y\in J}\sup\left\{\left(\delta\,|J| + \delta^{-\delta}\right)\left\|\int_y^{\cdot} f\right\|_{p+\delta} : f\in B_2\right\} \\
&= \left(\delta\,|J| + \delta^{-\delta}\right)^2 \inf_{y\in J}\sup\left\{\left\|\int_y^{\cdot} \frac{f}{\delta\,|J|+\delta^{-\delta}}\right\|_{p+\delta} : \left\|\frac{f}{\delta\,|J|+\delta^{-\delta}}\right\|_p \le 1\right\} \\
&= \left(\delta\,|J| + \delta^{-\delta}\right)^2 \inf_{y\in J}\sup\left\{\left\|\int_y^{\cdot} g\right\|_{p+\delta} : \|g\|_p \le 1\right\} \\
&= \left(\delta\,|J| + \delta^{-2}\right)^{-2} A_{p(\cdot),p(\cdot)+\delta}(J).
\end{aligned}
$$

The proof of the remaining part of the claimed inequality is similar. □

Lemma 9.13. *Let $J\subset I$ be an interval with $|J|\le 1$ and suppose that $\widetilde{p}\in(1,\infty)$. Then there is a bounded positive function η defined on $(0,1)$, with $\eta(\delta)\to 0$ as $\delta\to 0$, such that if $p,q\in\mathscr{P}(J)$ with*

$$\widetilde{p}\le p(x)\le \widetilde{p}+\delta,\ \widetilde{p}\le q(x)\le\widetilde{p}+\delta \text{ in } J,$$

then

$$(1-\eta(\delta))\,|J|^{2\delta} \le \frac{A_{p(\cdot)}(J)}{A_{q(\cdot)}(J)} \le (1+\eta(\delta))\,|J|^{-2\delta}.$$

Proof. We prove only the right-hand inequality as the rest follows in a similar fashion. By Lemma 9.6 and (9.12) we have

$$
\begin{aligned}
\frac{A_{p(\cdot)}(J)}{A_{q(\cdot)}(J)} &\le (\delta|J|+\delta^{-\delta})^4\frac{A_{\widetilde{p},\widetilde{p}+\delta}(J)}{A_{\widetilde{p}+\delta,\widetilde{p}}(J)} = \left(\delta\,|J|+\delta^{-\delta}\right)^4 \frac{\mathfrak{B}(\widetilde{p},\widetilde{p}+\delta)}{\mathfrak{B}(\widetilde{p}+\delta,\widetilde{p})}|J|^{-2\delta/(\widetilde{p}(\widetilde{p}+\delta))} \\
&\le (\delta|J|+\delta^{-\delta})^4\frac{\mathfrak{B}(\widetilde{p},\widetilde{p}+\delta)}{\mathfrak{B}(\widetilde{p}+\delta,\widetilde{p})}\,|J|^{-2\delta}.
\end{aligned}
$$

Since

$$\lim_{\delta\to 0}\left(\delta\,|J|+\delta^{-\delta}\right)^4\frac{\mathfrak{B}(\widetilde{p},\widetilde{p}+\delta)}{\mathfrak{B}(\widetilde{p}+\delta,\widetilde{p})} = 1,$$

the choice

$$\eta(\delta) = \max\left\{\delta, \left(\delta\,|J|+\delta^{-\delta}\right)^4\frac{\mathfrak{B}(\widetilde{p},\widetilde{p}+\delta)}{\mathfrak{B}(\widetilde{p}+\delta,\widetilde{p})} - 1\right\}$$

establishes the lemma. □

Lemma 9.14. *Let $p\in\mathscr{P}(I), \delta>0, a_1<b_1\le a_2<b_2$ and $J_i=(a_i,b_i)\subset I$ $(i=1,2)$; let f_1,f_2 be functions on I such that supp $f_i\subset J_i$ $(i=1,2)$ and $\|Tf_1\|_{p,J_1}>\delta$. Then*

$$\|T(f_1-f_2)\|_{p,I}>\delta.$$

Proof. Since $\|T(f_1/\delta)\|_{p,J_1} > 1$ we have

$$\int_{a_1}^{b_1} \left| \int_{a_1}^{x} \frac{f_1(t)}{\delta} \right|^{p(x)} dx > 1.$$

Thus

$$\begin{aligned}\int_a^b \left| \frac{T(f_1-f_2)(x)}{\delta} \right|^{p(x)} dx &= \int_a^b \left| \int_a^x \frac{f_1(t)-f_2(t)}{\delta} dt \right|^{p(x)} dx \\ &\geq \int_{a_1}^{b_1} \left| \int_a^x \frac{f_1(t)-f_2(t)}{\delta} dt \right|^{p(x)} dx \\ &= \int_{a_1}^{b_1} \left| \int_a^x \frac{f_1(t)}{\delta} dt \right|^{p(x)} dx > 1,\end{aligned}$$

and so $\|T(f_1-f_2)\|_{p,I} > \delta$. □

In what follows we shall need a restriction on the function $p \in \mathscr{P}(I)$ that is a little stronger than the log-Hölder condition (9.11). We remind the reader that $[a,b]$ is a compact interval.

Definition 9.4. A function $p \in \mathscr{P}(I)$ is said to be strongly log-Hölder continuous (written $p \in \mathscr{SLH}(I)$) if there is an increasing continuous function ψ defined on $[0,|I|]$ such that $\lim_{t\to 0+} \psi(t) = 0$ and

$$-|p(x)-p(y)|\log|x-y| \leq \psi(|x-y|) \text{ for all } x,y \in I \text{ with } 0 < |x-y| < 1/2. \tag{9.14}$$

It is easy to see that Lipschitz or Hölder functions belong to $\mathscr{SLH}(I)$.

Proposition 9.5. *Let $p \in \mathscr{SLH}(I)$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon N(\varepsilon) = \frac{1}{2\pi} \int_I \left(p'(x) p(x)^{p(x)-1} \right)^{1/p(x)} \sin(\pi/p(x))\, dx,$$

where N stands for $N_{A_{p(\cdot)}}$ or $N_{C_{p(\cdot)}}$.

Proof. We prove only the case $N = N_{A_{p(\cdot)}}$, the other case following in a similar manner. Let $N \in \mathbb{N}$. By Lemma 9.9, there are a constant $\varepsilon_N > 0$ and a set of non-overlapping intervals $\{I_i^N\}_{i=1}^N$ covering I such that $A_{p(\cdot)}(I_i^N) = \varepsilon_N$ for every i. Let q_N be the step-function defined by

$$q_N(x) = \sum_{i=1}^N p_{I_i^N}^+ \chi_{I_i^N}(x)$$

and set

$$\delta_{N,i} = p^{+}_{I_i^N} - p^{-}_{I_i^N}.$$

Then

$$p(x) \leq q_N(x) \leq p(x) + \delta_{N,i} \text{ for } i = 1, ..., N.$$

Claim 1 $\varepsilon_N \to 0$ as $N \to \infty$.

To prove this, note that clearly ε_N is non-increasing. Suppose that there exists $\delta > 0$ such that $\varepsilon_N > \delta$ for all N. Fix N and set $I_i^N = I_i = (a_i, a_{i+1})$. Since $A_{p(\cdot),I_i} > \delta$, for each $i \in \{1, ..., N\}$ there is a function f_i, with supp $f_i \subset I_i$, such that $\|f_i\|_{p,I_i} \leq 1$ and $\left\| \int_{a_i}^{\cdot} f_i \right\|_{p,I_i} = \|T f_i\|_{p,I_i} > \delta$. By Lemma 9.14,

$$\left\|T(f_i - f_j)\right\|_{p,I_i} > \delta \text{ for } i < j,$$

and so there are N functions $f_1, ..., f_N$ in the unit ball of $L_{p(\cdot),I}$ such that

$$\left\|T(f_i - f_j)\right\|_{p,I_i} > \delta \text{ for } i \neq j.$$

Since N can be arbitrarily large, this contradicts the compactness of T and establishes the claim.

Claim 2 $\lim_{N\to\infty} \max\left\{\left|I_i^N\right| : i = 1, 2, ..., N\right\} = 0.$

If this were false, there would be sequences $\{N_k, i_k\}_{k=1}^{\infty}$, $i_k \in \{1, 2, ..., N_k\}$, and an interval J such that $J \subset I_{i_k}^{N_k}$ for each k, so that

$$\varepsilon_{N_k} = A_{p(\cdot)}\left(I_{i_k}^{N_k}\right) \geq A_{p(\cdot)}(J) > 0,$$

contradicting the fact that $\varepsilon_N \to 0$.

Claim 3 There is a sequence $\{\beta_N\}$, with $\beta_N \downarrow 1$, such that for all $i \in \{1, ..., N\}$,

$$\beta_N^{-1} \varepsilon_N \left|I_i^N\right|^{2\delta_{N,i}} \leq A_{q_N(\cdot)}\left(I_i^N\right) \leq \beta_N \varepsilon_N \left|I_i^N\right|^{-2\delta_{N,i}}.$$

To establish this, note that since $p^- \leq q_N(x), p(x) \leq p^- + \delta_{N,i}$ on I_i^N we have, by Lemma 9.13,

$$(1 - \eta(\delta_{N,i}))\left|I_i^N\right|^{2\delta_{N,i}} \leq \frac{A_{p(\cdot)}\left(I_i^N\right)}{A_{q_N(\cdot)}\left(I_i^N\right)} \leq (1 + \eta(\delta_{N,i}))\left|I_i^N\right|^{-2\delta_{N,i}}.$$

Using $\varepsilon_N = A_{p(\cdot)}\left(I_i^N\right)$ this gives

$$\frac{\varepsilon_N}{1 + \eta(\delta_{N,i})}\left|I_i^N\right|^{2\delta_{N,i}} \leq A_{q_N(\cdot)}\left(I_i^N\right) \leq \frac{\varepsilon_N}{1 - \eta(\delta_{N,i})}\left|I_i^N\right|^{-2\delta_{N,i}},$$

and the claim follows.

Claim 4 For all N and all $i \in \{1,2,...,N\}$,

$$\left|I_i^N\right|^{-\delta_{N,i}} \leq e^{\psi\left(\left|I_i^N\right|\right)}.$$

Fix I_i^N. The function p is continuous on I since it belongs to $\mathscr{SLH}(I)$. As $p_{I_i^N}^+ - p_{I_i^N}^- = \delta_{N,i}$, there are points $x,y \in I_i^N$ with $|p(x)-p(y)| = \delta_{N,i}$. Using (9.14) we obtain

$$\left|I_i^N\right|^{-\delta_{N,i}} \leq |x-y|^{-|p(x)-p(y)|} \leq e^{\psi(|x-y|)} \leq e^{\psi\left(\left|I_i^N\right|\right)}.$$

Claim 5 There is a constant $C > 0$ such that for all N and all $i \in \{1,2,...,N\}$,

$$C^{-1}\varepsilon_N \leq \left|I_i^N\right| \leq C\varepsilon_N.$$

Since $q_N = p_{I_i^N}^- + \delta_{N,i} := r_{N,i}$ is a constant function on I_i^N, by Lemma 9.6 we have

$$A_{q_N(\cdot)}\left(I_i^N\right) = \mathfrak{B}\left(r_{N,i}, r_{N,i}\right)\left|I_i^N\right|.$$

It is easy to see that there exists $a > 0$ such that $a^{-1} \leq \mathfrak{B}\left(r_{N,i}, r_{N,i}\right) \leq a$ for all N and all $i \in \{1,2,...,N\}$. Using Claim 4 we see that

$$\left|I_i^N\right|^{-2\delta_{N,i}} \leq e^{2\psi\left(\left|I_i^N\right|\right)} \leq e^{2\psi(|I|)} := K,$$

and by Claim 3,

$$K^{-1}\beta_N^{-1}\varepsilon_N \leq \mathfrak{B}\left(r_{N,i}, r_{N,i}\right)\left|I_i^N\right| \leq K\beta_N\varepsilon_N.$$

Hence

$$a^{-1}K^{-1}\beta_N^{-1}\varepsilon_N \leq \left|I_i^N\right| \leq aK\beta_N\varepsilon_N.$$

Since $\beta_N \to 1$ as $N \to \infty$, the claim follows.

Having justified these various claims we can now proceed to finish the proof of the proposition. Since by Claim 1, $\varepsilon_N \to 0$ we know, by Claim 5, that

$$\max\left\{\left|I_i^N\right| : i = 1,...,N\right\} \to 0$$

as $N \to \infty$; and by Claim 4,

$$\left|I_i^N\right|^{-2\delta_{N,i}} \leq e^{2\psi\left(\left|I_i^N\right|\right)} \leq e^{2\psi\left(\max\left\{\left|I_i^N\right| : i=1,...,N\right\}\right)} := \gamma_N \to 1.$$

Put $\alpha_N = \beta_N\gamma_N$: then $\alpha_N \to 1$ and, by Claim 2,

$$\alpha_N^{-1}\varepsilon_N \leq A_{q_N(\cdot)}\left(I_i^N\right) \leq \alpha_N\varepsilon_N. \tag{9.15}$$

Moreover, by Claim 5 we have

$$N\varepsilon_N = C\sum_{i=1}^{N} C^{-1}\varepsilon_N \le C\sum_{i=1}^{N} \left|I_i^N\right| = C\left|I\right|,$$

which gives, by (9.15),

$$\begin{aligned} N\varepsilon_N\left(\alpha_N^{-1}\varepsilon_N - 1\right) &= \sum_{i=1}^{N}\left(\alpha_N^{-1}\varepsilon_N - \varepsilon_N\right) \le \sum_{i=1}^{N} A_{q_N(\cdot)}\left(I_i^N\right) - N\varepsilon_N \\ &\le \sum_{i=1}^{N}\left(\alpha_N\varepsilon_N - \varepsilon_N\right) = N\varepsilon_N\left(\alpha_N - 1\right). \end{aligned}$$

Thus

$$\left|\sum_{i=1}^{N} A_{q_N(\cdot)}\left(I_i^N\right) - N\varepsilon_N\right| \to 0 \text{ as } N\to\infty.$$

On the other hand we have, by Theorem 5.8 (recall again that q_N is constant on I_i^N),

$$\begin{aligned} \sum_{i=1}^{N} A_{q_N(\cdot)}\left(I_i^N\right) &= \frac{1}{2\pi}\sum_{i=1}^{N}\left(q_N'(\cdot)q_N(\cdot)^{q_N(\cdot)-1}\right)^{1/q_N(\cdot)}\sin\left(\pi/q_N(\cdot)\right)\left|I_i^N\right| \\ &\to \frac{1}{2\pi}\int_I\left(p'(x)p(x)^{p(x)-1}\right)^{1/p(x)}\sin(\pi/p(x))dx, \end{aligned}$$

and so

$$\lim_{N\to\infty} N\varepsilon_N = \frac{1}{2\pi}\int_I\left(p'(x)p(x)^{p(x)-1}\right)^{1/p(x)}\sin(\pi/p(x))dx.$$

Since ε_N depends monotonely on N it is not difficult to see that $\lim_{N\to\infty} N\varepsilon_N = \lim_{\varepsilon\to 0}\varepsilon N(\varepsilon)$, and consequently

$$\lim_{\varepsilon\to 0}\varepsilon N(\varepsilon) = \frac{1}{2\pi}\int_I\left(p'(x)p(x)^{p(x)-1}\right)^{1/p(x)}\sin(\pi/p(x))dx.$$

The proof is complete. □

Given $p\in\mathscr{SLH}(I)$, we construct step-functions that are approximations to p. Let $N\in\mathbb{N}$ and use Lemma 9.9, applied to the function $D := D_{p(\cdot)}$: there exists $\varepsilon > 0$ such that $N_D(\varepsilon) = N$and there are non-overlapping intervals I_i^D $(i = 1,\ldots,N)$ that cover I and are such that $D(I_i^D) = \varepsilon$ for $i = 1,\ldots,N$. Define

$$p_{D,N}^{+}(x) = \sum_{i=1}^{N} p_{I_i^D}^{+}\chi_{I_i^D}(x),\; p_{D,N}^{-}(x) = \sum_{i=1}^{N} p_{I_i^D}^{-}\chi_{I_i^D}(x);$$

step-functions $p^+_{B,N}$ and $p^-_{B,N}$ are defined in an exactly similar way, with the function B in place of D and with intervals I_i^B arising from the use of that part of Lemma 9.9 related to B.

Lemma 9.15. *Let $p \in \mathscr{P}(I)$ and $N \in \mathbb{N}$. Let $\varepsilon > 0$ correspond to N in the sense of Lemma 9.9, applied to B, so that $N_B(\varepsilon) = N$, and write*

$$p^-(x) = p^-_{B,N}(x),\ p^+(x) = p^+_{B,N}(x),$$

where $p^+_{B,N}$ and $p^-_{B,N}$ are defined as indicated above. Then

$$a_{N+1}\left(T : L_{p^-(\cdot)}(I) \to L_{p^+(\cdot)}(I)\right) \leq \varepsilon.$$

Proof. In the notation of Lemma 9.9, there are intervals I_i^B such that $B(I_i^B) = \varepsilon$ for $i = 1, ..., N$. For each i there exists $y_i \in I_i^B$ such that

$$B(I_i^B) = \sup \left\{ \left\| \int_{y_i}^{\cdot} f \right\|_{p^+, I_i^B} : \|f\|_{p^-, I_i^B} \leq 1 \right\}.$$

Define

$$P_N f(x) = \sum_{i=1}^{N} \int_a^{y_i} f(y)dy \cdot \chi_{I_i^B}(x);$$

plainly P_N has rank N. Let $f \in L_{p^-(\cdot)}(I)$ and set

$$\lambda_0 = \varepsilon \|f\|_{p^-, I}. \tag{9.16}$$

Then

$$1 = \int_I \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p^-(x)} dx = \sum_{i=1}^{N} \int_{I_i^B} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p^-(x)} dx.$$

Recall that on I_i^B the functions p^- and p^+ have constant values p_i^-, p_i^+, say, respectively, with $p_i^+/p_i^- \geq 1$. Thus

$$1 \geq \sum_{i=1}^{N} \left(\int_{I_i^B} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p_i^-} dx \right)^{p_i^+/p_i^-} = \sum_{i=1}^{N} (\varepsilon/\lambda_0)^{p_i^+} \left(\int_{I_i^B} |f(x)|^{p_i^-} dx \right)^{p_i^+/p_i^-}.$$

Use of the fact that

$$\varepsilon = \sup_f \left(\int_{I_i^B} \left| \int_{y_i}^{x} f(y)dy \right|^{p_i^+} dx \right)^{1/p_i^+} / \left(\int_{I_i^B} |f(y)|^{p_i^-} dy \right)^{1/p_i^-}$$

now gives

$$1 \geq \sum_{i=1}^{N} (1/\lambda_0)^{p_i^+} \int_{I_i^B} \left| \int_{y_i}^{x} f(y)dy \right|^{p_i^+} dx = \sum_{i=1}^{N} \int_{I_i^B} \left| \frac{\int_{y_i}^{x} f(y)dy}{\lambda_0} \right|^{p_i^+} dx$$
$$= \int_I \left| \frac{(T-P_N)(f)(x)}{\lambda_0} \right|^{p^+(x)} dx,$$

from which it follows that $\|(T-P_N)f\|_{p^+,I} \leq \lambda_0$. Using the definition (9.16) of λ_0 we see that

$$\|(T-P_N)f\|_{p^+,I} \leq \varepsilon \|f\|_{p^-,I},$$

and so $a_{N+1}\left(T : L_{p^-}(I) \to L_{p^+}(I)\right) \leq \varepsilon$, as claimed. □

We next obtain a lower estimate for the Bernstein numbers.

Lemma 9.16. *Let $p \in \mathscr{P}(I)$ and $N \in \mathbb{N}$. Let $\varepsilon > 0$ correspond to N in the sense of Lemma 9.9, applied to D, so that $N_D(\varepsilon) = N$, and write*

$$p^-(x) = p^-_{D,N}(x),\ p^+(x) = p^+_{D,N}(x),$$

where $p^+_{D,N}$ and $p^-_{D,N}$ are defined as indicated above. Then

$$b_N\left(T : L_{p^+(\cdot)}(I) \to L_{p^-(\cdot)}(I)\right) \geq \varepsilon.$$

Proof. In the notation of Lemma 9.9, there are intervals I_i^D such that $D(I_i^D) = \varepsilon$ for $i = 1, ..., N$. Since T is compact, for each i there exists $f_i \in L_{p^+(\cdot)}(I_i^D)$, with supp $f_i \subset I_i^D$, such that

$$\|Tf_i\|_{p^-,I_i^D} / \|f_i\|_{p^+,I_i^D} = \varepsilon, \tag{9.17}$$

and $Tf_i(c_i) = Tf_i(c_{i+1}) = 0$, where c_i and c_{i+1} are the endpoints of I_i^D. On each I_i^D the functions p^- and p^+ are constant; denote these constant values by p_i^- and p_i^+, respectively and note that $p_i^- / p_i^+ \leq 1$. Set

$$X_N = \left\{ f = \sum_{i=1}^{N} \alpha_i f_i : \alpha_i \in \mathbb{R} \right\}.$$

Thus $\dim X_N = N$. Choose any non-zero $f \in X_N$ and put $\lambda_0 = \varepsilon \|f\|_{p^+,I}$. Then

$$1 = \int_I \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p^+(x)} dx = \sum_{i=1}^{N} \int_{I_i^D} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p_i^+(x)} dx$$
$$\leq \sum_{i=1}^{N} \left(\int_{I_i^D} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p_i^+(x)} dx \right)^{p_i^-/p_i^+}$$

$$= \sum_{i=1}^{N} (\varepsilon/\lambda_0)^{p_i^-} \left(\int_{I_i^D} |f(x)|^{p_i^+(x)}\, dx \right)^{p_i^-/p_i^+}$$
$$= \sum_{i=1}^{N} (\varepsilon/\lambda_0)^{p_i^-} \left(\int_{I_i^D} |\alpha_i f_i(x)|^{p_i^+}\, dx \right)^{p_i^-/p_i^+}.$$

Use of (9.17) now shows that

$$1 \le \sum_{i=1}^{N} (1/\lambda_0)^{p_i^-} \int_{I_i^D} |T(\alpha_i f_i)(x)|^{p_i^-}\, dx$$
$$= \sum_{i=1}^{N} \int_{I_i^D} \left| \frac{Tf(x)}{\lambda_0} \right|^{p_i^-} dx = \int_I \left| \frac{Tf(x)}{\lambda_0} \right|^{p^-(x)} dx,$$

from which it follows that

$$\varepsilon \le b_N \left(T : L_{p^+(\cdot)}(I) \to L_{p^-(\cdot)}(I) \right),$$

and the proof is complete. □

Theorem 9.5. *Let $p \in \mathscr{P}(I)$ be continuous on I. For all $N \in \mathbb{N}$ denote by ε_N numbers satisfying $N = N_B(\varepsilon_N)$. Then there are sequences K_N, L_N, with $K_N \to 1$ and $L_N \to 1$ as $N \to \infty$, such that:*

(i) $a_{N+1}(T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)) \le K_N \varepsilon_N$
(ii) $b_N(T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)) \ge L_N \varepsilon_N$

Proof. Because of the multiplicative property (S3) of the approximation numbers, $a_{N+1}(T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I))$ is majorised by

$$\left\| id_N^- : L_{p(\cdot)}(I) \to L_{p_{B,N}^-(\cdot)}(I) \right\|$$
$$\times\, a_{N+1}(T : L_{p_{B,N}^-(\cdot)}(I) \to L_{p_{B,N}^+(\cdot)}(I)) \,\times\, \left\| id_N^+ : L_{p_{B,N}^+(\cdot)}(I) \to L_{p(\cdot)}(I) \right\|,$$

where id_N^- and id_N^+ are the obvious embedding maps, while $p_{B,N}^+, p_{B,N}^-$ are the same as in Lemma 9.15, as is I_i^B, to be used next. Since $\left| I_i^B \right| \to 0$ when $N \to \infty$, and p is continuous, it is clear that

$$\left\| p - p_{B,N}^- \right\|_{\infty,I} \to 0 \text{ and } \left\| p - p_{B,N}^+ \right\|_{\infty,I} \to 0.$$

Thus by Corollary 9.2,

$$\left\| id_N^- : L_{p(\cdot)}(I) \to L_{p_{B,N}^-(\cdot)}(I) \right\| \to 1 \text{ and } \left\| id_N^+ : L_{p_{B,N}^+(\cdot)}(I) \to L_{p(\cdot)}(I) \right\| \to 1$$

as $N \to \infty$. Part (i) now follows from Lemma 9.15. The proof of (ii) is similar, with the aid this time of Lemma 9.16. □

Theorem 9.6. *Let $p \in \mathscr{SLH}(I)$. Then*

$$\lim_{n\to\infty} ns_n(T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)) = \frac{1}{2\pi}\int_I \left\{p'(t)p(t)^{p(t)-1}\right\}^{1/p(t)} \sin\left(\pi/p(t)\right) dt,$$

where s_n denotes the nth approximation, Gelfand, Kolmogorov or Bernstein number of T.

Proof. Use Theorem 9.5, Proposition 9.5 and the inequalities of Theorems 5.4 and 5.5. □

The proofs of Theorems 9.4 and 9.6 may be combined to give the following theorem, which contains both these results.

Theorem 9.7. *Let I be representable as the finite union of non-overlapping intervals J_i $(i = 1,...,m)$ and suppose that $p \in \mathscr{SLH}(I_i)$ for each $i \in \{1,2,...,m\}$. Then*

$$\lim_{n\to\infty} ns_n(T : L_{p(\cdot)}(I) \to L_{p(\cdot)}(I)) = \frac{1}{2\pi}\int_I \left\{p'(t)p(t)^{p(t)-1}\right\}^{1/p(t)} \sin\left(\pi/p(t)\right) dt,$$

where s_n denotes the nth approximation, Gelfand, Kolmogorov or Bernstein number of T.

9.3 A Version of the p-Laplacian

Throughout this section we shall suppose that Ω is a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (that is, C^∞) boundary and that $p \in \mathscr{P}(\Omega)$. The first-order Sobolev space modelled on $L_{p(\cdot)}(\Omega)$ is defined to be

$$W^1_{p(\cdot)}(\Omega) = \left\{u \in L_{p(\cdot)}(\Omega) : |\nabla u| \in L_{p(\cdot)}(\Omega)\right\} \quad \left(|\nabla u|^2 = \sum_{i=1}^n |D_i u|^2\right),$$

endowed with the norm

$$\|u\| := \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}. \tag{9.18}$$

The closure of $C_0^\infty(\Omega)$ in $W^1_{p(\cdot)}(\Omega)$ is denoted by $\overset{0}{W}{}^1_{p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$, then by the Poincaré inequality the norm given in (9.18) is equivalent on $\overset{0}{W}{}^1_{p(\cdot)}(\Omega)$ to

$$\|u\|_{1,p,\Omega} := \|\nabla u\|_{p,\Omega}, \tag{9.19}$$

(see [81]) and we shall henceforth suppose that $\overset{0}{W}{}^1_{p(\cdot)}(\Omega)$ is endowed with this equivalent norm.

In [37] it is shown that whenever $p \in \mathscr{P}(\Omega)$, the space $L_{p(\cdot)}(\Omega)$ is smooth (see Remark 1.2) and the gradient of its norm at any point $u \neq 0$ is given by

$$\left\langle h, \text{grad } \|u\|_{p,\Omega} \right\rangle_{L_{p(\cdot)}(\Omega)} = \frac{\int_\Omega p(x)\,|u(x)|^{p(x)-2}\,u(x)\,\|u\|_{p,\Omega}^{-p(x)}\,h(x)dx}{\int_\Omega p(x)\,|u(x)|^{p(x)}\,\|u\|_{p,\Omega}^{-p(x)-1}\,dx} \tag{9.20}$$

for all $h \in L_{p(\cdot)}(\Omega)$. This is a key step in the proof, again given in [37], of the following theorem.

Theorem 9.8. *Let $p \in C(\overline{\Omega})$ and write $X := \overset{0}{W}{}^1_{p(\cdot)}(\Omega)$. Then:*

(i) X is smooth and the gradient of its norm at any point $u \neq 0$ is given by

$$\left\langle h, grad\ \|u\|_{1,p,\Omega} \right\rangle_X = \frac{\int_\Omega p(x)\,|\nabla u(x)|^{p(x)-2}\,\|u\|_{1,p,\Omega}^{-p(x)}\,\nabla u(x)\cdot\nabla h(x)dx}{\int_\Omega p(x)\,|\nabla u(x)|^{p(x)}\,\|u\|_{1,p,\Omega}^{-p(x)-1}\,dx} \tag{9.21}$$

for all $h \in X$.

(ii) If $p(x) \geq 2$ for all $x \in \overline{\Omega}$, then X is uniformly convex.

From these results a formula for duality maps on $\overset{0}{W}{}^1_{p(\cdot)}(\Omega)$ is obtained in [37]. More precisely, the following is established.

Theorem 9.9. *Let $p \in C(\overline{\Omega})$ be such that $p(x) \geq 2$ for all $x \in \overline{\Omega}$ and let J_ϕ be a duality map on $X := \overset{0}{W}{}^1_{p(\cdot)}(\Omega)$ with gauge function ϕ. Then for all $u, h \in X$ with $u \neq 0$,*

$$\langle h, J_\phi u\rangle_X = \frac{\phi\left(\|u\|_{1,p,\Omega}\right)\int_\Omega p(x)\,|\nabla u(x)|^{p(x)-2}\,\|u\|_{1,p,\Omega}^{-p(x)}\,\nabla u(x)\cdot\nabla h(x)dx}{\int_\Omega p(x)\,|\nabla u(x)|^{p(x)}\,\|u\|_{1,p,\Omega}^{-p(x)-1}\,dx}.$$

Following [37] we now give a version of the classical p-Laplacian appropriate, from the standpoint of duality maps, for the case of variable p. Namely, when $p \in C(\overline{\Omega})$ and J_ϕ is a duality map on $X := \overset{0}{W}{}^1_{p(\cdot)}(\Omega)$ with gauge function ϕ, the $(\phi, p(\cdot))$-Laplacian is defined to be the map $\Delta_{(\phi,p(\cdot))} : X \to X^*$ given by $\Delta_{(\phi,p(\cdot))} = -J_\phi$. Use of Proposition 1.23 now shows that there exists $u_1 \in X$, with $\|u_1\|_X = 1$, and $\mu_1 > 0$, such that

$$\int_\Omega p(x)\,|\nabla u_1(x)|^{p(x)-2}\,\nabla u_1(x)\cdot\nabla v(x)dx = \mu_1 \int_\Omega \frac{p(x)\,|u_1(x)|^{p(x)-2}\,u_1(x)v(x)}{\|u_1\|_{p,\Omega}^{p(x)}}dx$$

for all $v \in X$. This means that u_1 is a weak solution of the Dirichlet problem

$$\sum_{i=1}^{n} D_i \left(p(x) \, |\nabla u(x)|^{p(x)-2} D_i u(x) \right) = -\mu_1 p(x) \, |u(x)|^{p(x)-2} u(x) \, \|u\|_{p,\Omega}^{-p(x)} \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

The existence of a sequence of k-weak solutions of this problem follows just as in the discussion of the classical p-Laplacian given in Chap. 3.

Notes

Note 9.1. The standard reference for the basic properties of variable exponent spaces is the 1991 paper by Kováčik and Rákosník [81]. For a fairly comprehensive account of the current picture, see [36].

Note 9.2. The material in this section is based on the paper [54]. No other material on the s-numbers of Hardy operators acting on variable exponent spaces seems to be available at this time.

Note 9.3. The literature on various forms of the $p(x)$-Laplacian is quite large. The work of [37] which plays a crucial rôle in this section has the advantage that the necessary duality map can be calculated: it is the need for this that determines the particular form of the $p(x)$-Laplacian that we study.

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Edited by J.-M. Morel, B. Teissier, P.K. Maini

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